Solutions to the exam in Nonlinear Control
2E1262, 2004-12-17

1. (a) The assumptions on $f(x)$ can be written $xf(x) < 0$, $x \neq 0$. Now take the radially unbounded Lyapunov function $V(x) = x^2/2$ for which

$$\dot{V} = xf(x) < 0, \quad x \neq 0$$

Hence, Lyapunov Stability Theorem shows that $x = 0$ is globally asymptotically stable.

(b)

$$\begin{aligned} \frac{d}{dt}\tan(t) &= \frac{1}{\cos^2(t)} \\ 1 + \tan^2(t) &= 1 + \frac{\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)} \\ \tan(0) &= 0 \end{aligned}$$

and hence $x(t) = \tan(t)$ satisfies the differential equation

$$\dot{x}(t) = 1 + x^2(t), \quad x(0) = 0$$

Now $\lim_{t\to\pi/2} \tan(t) = \infty$, *i.e.*, the solution has finite escape time.

(c)

$$\begin{aligned} \dot{y} &= 2 + \sin x_1(t) \\ \ddot{y} &= (\cos x_1(t))\dot{x}_1 = (\cos x_1(t))u(t) \end{aligned}$$

Now take

$$u(t) = \frac{1}{(\cos x_1(t))}(r(t) - 2\dot{y}(t) - y(t))$$

to give the closed loop system $\ddot{y}(t) + 2\dot{y}(t) + y(t) = r(t)$. Finally express $u(t)$ in the state variables

$$u(t) = \frac{1}{(\cos x_1(t))}(r(t) - 4 - 2\sin x_1(t) - x_2(t))$$

2. The quantizer can be viewed as the sum of two deadzones with relays

$$f(e) = f_1(e) + f_2(e)$$

The describing function for the deadzone with relay

$$f(e) = \begin{cases} 1 & e > D \\ 0 & |e| \leq D \\ -1 & e < -D \end{cases}$$

equals

$$N(A) = \begin{cases} 0 & A < D \\ \frac{4}{A\pi}\sqrt{(1-(D/A)^2} & A \geq D \end{cases}$$

The deadzone for the first is $D = 0.5$ and gain $\alpha = 1$. For the second we have $D = 2$ and $\alpha = 3$.

Using gain formula $N_{\alpha f}(A) = \alpha N_f(A)$ and the summation formula $N_f(A) = N_{f_1}(A) + N_{f_2}(A)$, we obtain the answer

$$N_f(A) = \begin{cases} 0 & A < 1/2 \\ \frac{4}{A\pi}\sqrt{(1-(1/2A)^2} & 1/2 \leq A \leq 2 \\ \frac{4}{A\pi}\sqrt{(1-(1/2A)^2} + \frac{12}{A\pi}\sqrt{(1-(2/A)^2} & A \geq 2 \end{cases}$$

3. (a) The function $V(x) = x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_1x_3$ satisfies: $V(0) = 0$, It is strictly positive $V(x) > 0$, $x \neq 0$, since

$$V(x) = x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_1x_3 = (x_1 + \frac{1}{2}x_3)^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^2$$

and radially unbounded.

$$\begin{aligned} \dot{V} &= (2x_1 + x_3 \; x_2 \; x_3 + x_1)(-x_2 - (2x_1(t) + x_3(t))^3 \; x_1, \; x_2)^T \\ &= -2x_1x_2 - x_2x_3 - (2x_1(t) + x_3(t))^4 + x_1x_2 + x_1x_2 + x_2x_3 \\ &= -(2x_1(t) + x_3(t))^4 \leq 0 \end{aligned}$$

This implies that that the closed loop system is locally stable for $x = 0$.

Using LaSalle's invariant theorem it is possible to show that the system is locally asymptotically stable. The set E is the set $x_3 + 2x_1 = 0$. Then

$$\frac{d}{dt}(x_3 + 2x_1) = \dot{x}_3 + 2\dot{x}_1 = x_2 - 2x_2 = -x_2$$

For invariance we thus need $x_2 = 0$. Now $\dot{x_2} = x_1$ and hence $x_1 = 0$ and also $x_3 = 0$. Hence the largest invariant set in E is $x = 0$.

(b) The A-matrix of the linearized system equals

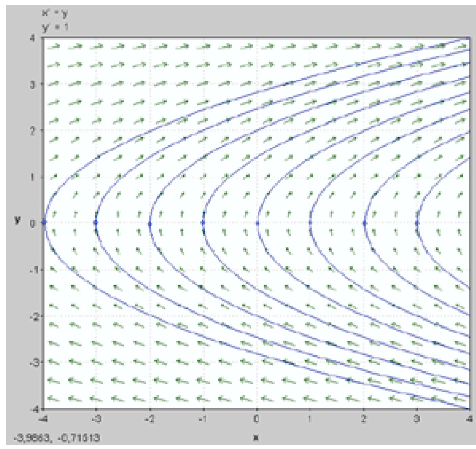
$$A = \begin{pmatrix} -1 + A_m & -1 & 1 - A_m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and hence $\det(sI - A) = s^3 + (1 - A_m)s^2 + s + (A_m - 1)$. This polynomial will have one negative coefficient $A_m \neq 1$. Hence $x = 0$ is an unstable equilibrium point when $A_m \neq 1$. For $A_m = 1$ the linearized system will have eigenvalues with real part zero and nothing can be said. However, in (a) it has been shown that the nonlinear system is asymptotically stable.

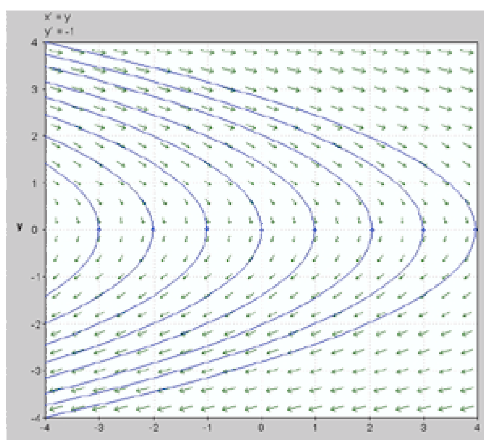
4. (a) Solving the differential equations for $u(t) = 1$ we have

$$x_1(t) = C_1 + C_2 t + t^2/2$$
$$x_2(t) = C_2 + t.$$

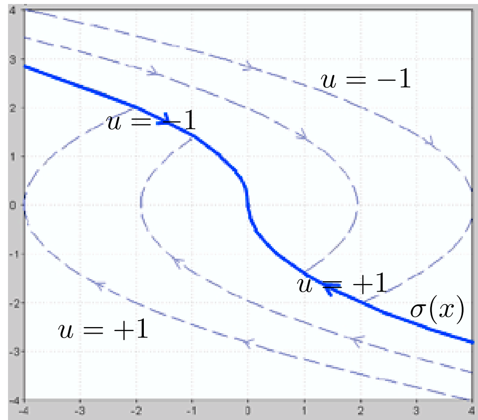
Eliminating the variable t we get $x_1(t) = \frac{1}{2}x_2^2(t) + C_3$, where $C_3 = C_1 - C_2^2/2$ is constant. These are parabolas (with x_1 on the x-axis and x_2 on the y-axis) as shown in the following figure:



Similarly for $u = -1$ we get $x_1(t) = -\frac{1}{2}x_2^2(t) + C_4$, where $C_4 = C_1 + C_2^2/2$ is a constant. These are parabolas as shown in the following figure:



(b) The control strategy is shown in the following figure:



With solid line is shown the switching surface. With dashed line are shown some trajectories.

(c) The optimal control strategy requires at most one switch. We have a switch only when the initial condition is not on the surface $\sigma(x)$. When the coordinate $x_2(t) < 0$ then the trajectory follows a parabola with equation $x_1 = \frac{x_2^2}{2} + C_3$ and the input is $u = +1$. If $x_2(t) > 0$ the the trajectory is the parabola $x_1 = -\frac{x_2^2}{2} + C_4$ and the input is $u = -1$, as shown in the previous figure. The switching surface has equation

$$\sigma(x) = \begin{cases} x_1 - \frac{x_2^2}{2} = 0, & \text{if } x_2 < 0; \\ x_1 + \frac{x_2^2}{2} = 0, & \text{if } x_2 > 0, \end{cases}$$

or in a compact form $\sigma(x) = x_1 + \text{sign}\{x_2\}\frac{x_2^2}{2}$ Thus the input $u(x(t))$ can be written as $u = -\text{sign}\{\sigma(x)\}$.

(d) The initial condition $x(0) = (2, -2)^T$ is a point on the switching surface $\sigma(x)$, since $\sigma\big(x(0)\big) = 2 + (-1)\frac{2^2}{2} = 0$. The optimal input is $u(t) = +1$ as shown in (c). Let t_f be the final time, then from (a) we have

$$\begin{aligned} x_1(t_f) &= 0 = C_1 + C_2 t_f + t_f^2/2 \\ x_2(t_f) &= 0 = C_2 + t_f \end{aligned}$$

where $C_1 = x_1(0) = 2$ and $C_2 = x_2(0) = -2$. We immediately get that $t_f = -C_2 = 2$.

If the initial condition is $x = (2, -2-\epsilon)^T$ then this means that there will be a switch. Let the total time it takes to drive the system from the initial condition to zero be $T_{tot} = T_1 + T_2$, where T_1 is the time to drive the system from the initial condition to the switching surface, and T_2 is the time it takes to drive the system to zero after the switch.

Let us compute T_1:
Since $x_2(0) < 0$ then this means $u = +1$ until we arrive on $\sigma(x)$. Thus the system is following the trajectory

$$
\begin{aligned}
x_1(t) &= C_1 + C_2 t + \frac{t^2}{2} \\
x_2(t) &= C_2 + t
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= x_1(0) = 2 \\
C_2 &= x_2(0) = -2 - \epsilon.
\end{aligned}
$$

The time T_1 is the time when the trajectory starting at $x(0) = (2, -2-\epsilon)^T$ intersects the switching surface $\sigma(x)$. The we have a system of equations

$$
\begin{aligned}
x_1(T_1) &= \frac{x_2^2(T_1)}{2} + C_1 - \frac{C_2^2}{2} \\
x_1(T_1) &= -\frac{x_2^2(T_1)}{2}
\end{aligned}
$$

from where we get

$$
\begin{aligned}
x_2^2(T_1) &= \frac{C_2^2}{2} - C_1 \Rightarrow x_2(T_1) = \sqrt{\frac{C_2^2}{2} - C_1} = \sqrt{\frac{\epsilon^2}{2} + \epsilon} \\
x_1(T_1) &= -\frac{1}{2}\left(\frac{\epsilon^2}{2} + \epsilon\right).
\end{aligned}
$$

Clearly we chose $x_2(T_1)$ positive since the switching surface from $u = +1$ to $u = -1$ is defined for $x_2 > 0$ (see figure in part (c)). Finally to compute the time T_1 we use the fact that

$$
\begin{aligned}
x_2(T_1) = C_2 + T_1 \Rightarrow T_1 = x_2(T_1) - C_2 &= \sqrt{\frac{\epsilon^2}{2} + \epsilon} - (-2-\epsilon) \\
&= 2 + \epsilon + \sqrt{\frac{\epsilon^2}{2} + \epsilon}
\end{aligned}
$$

Let us now compute T_2:
the system have reached at time T_1 the switching surface. Thus the control switches from $u = +1$ to $u = -1$ and the system evolves on the $\sigma(x)$ until it reaches 0. The time T_2 when the system reaches 0 can be simply computed from $0 = x_2(T_2) = \bar{C}_2 - T_2$ since this equation is valid when the system is driven by $u = -1$.

The constant C_2 depends on the initial condition after the switching, namely $\bar{C}_2 = x_2(T_1) = \sqrt{\frac{\epsilon^2}{2} + \epsilon}$. Thus $T_2 = \bar{C}_2$ and thus

$$
\begin{aligned}
T_{tot} &= \underbrace{2 + \epsilon + \sqrt{\frac{\epsilon^2}{2} + \epsilon}}_{T_1} + \underbrace{\sqrt{\frac{\epsilon^2}{2} + \epsilon}}_{T_2} \\
&= 2 + \epsilon + 2\sqrt{\frac{\epsilon^2}{2} + \epsilon}.
\end{aligned}
$$

Clearly for $\epsilon = 0$, $T_{tot} = t_f = 2$ found in the first part of (d).

5. This problem can be written

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -2x_2(t) + u(t) \\
y(t) &= x_1(t)
\end{aligned}
$$

with the nonlinear static feedback $u = -f(y)$. The transfer function of the state space model equals

$$G(s) = \frac{1}{s(s+2)}$$

The nonlinear function corresponds to a circle with center at -1 and radius $r = 1/R$. Now the circle criterion gives stability if

$$|G(i\omega) - (-1)| > 1/R, \quad \forall \omega.$$

which implies that the sensitivity function $S = 1/(1+G)$ satisfies

$$|S(i\omega)| = \frac{1}{|1 + G(i\omega)|} < R \quad \forall \omega$$

Now

$$\left(\frac{1}{|1 + G(i\omega)|}\right)^2 = \frac{\omega^4 + 4\omega^2}{(1 + \omega^2)^2}$$

The derivative with respect to ω^2 equals

$$\frac{(4 - 2\omega^2)}{1 + \omega^2}$$

which is zero for $\omega = \sqrt{2}$. This gives that

$$\max_{\omega} |S(i\omega)| = \frac{2}{\sqrt{3}}.$$

Hence, The system is stable for $R > \frac{2}{\sqrt{3}}$.