Solutions to the exam in Nonlinear Control
2E1262, 2005-03-17

1. (a) For small x, $xf(x) \approx f(0)x + f'(x)x^2 < 0$. Now take the Lyapunov function $V(x) = x^2/2$, for which

$$\dot{V} = xf(x) < 0, \quad \text{for small } x$$

Hence, Lyapunov Stability Theorem shows that $x = 0$ is an asymptotically stable equilibrium. The result also follows directly from that the linearized system around $x = 0$ is asymptotically stable.

(b)

$$\begin{aligned} dt &= \frac{-1}{x^2}dx \quad \Rightarrow \quad t = \frac{1}{x} - \frac{1}{x_0} \quad \Rightarrow \\ x(t) &= \frac{1}{t + 1/x_0} \end{aligned}$$

For $x_0 = 0.01 > 0$ the solution tends to zero as $t \to \infty$, while for $x_0 = -0.01 < 0$ we have a finite escape time at $t = 100$.

(c) Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -h(x_1) + u(t) \end{aligned}$$

The condition $zh(z) > 0 \Rightarrow h(z) > 0$ for $z > 0$ and $h(z) < 0$ for $z < 0$, which implies that

$$\int_0^{x_1} h(z)dz > 0, \quad x_1 \neq 0.$$

Hence $V(0) = 0$ and $V(x) > 0$. Furthermore

$$\begin{aligned} \dot{V}(x) &= h(x_1)\dot{x}_1 + x_2\dot{x}_2 \\ &= h(x_1)x_2 + x_2(-h(x_1) - \sigma(x_2)) = -x_2\sigma(x_2) \end{aligned}$$

Hence $\dot{V}(x) \leq 0$ if $x_2\sigma(x_2) \geq 0 \ \forall x_2 \neq 0$, and the closed loop system is stable.

2. (a) The describing function for an odd nonlinearity is given by

$$N(A) = \frac{2}{\pi A}\int_0^{\pi} f(A\sin\phi)\sin\phi\, d\phi$$

In the integration interval $\sin\phi \geq 0$. From the condition on f it then follows that $k_1 A \sin\phi \leq f(A\sin\phi) \leq k_2 A \sin\phi$. Hence,

$$N(A) \geq \frac{2}{\pi A}\int_0^\pi k_1 A \sin\phi \sin\phi \, d\phi = k_1$$
$$N(A) \leq \frac{2}{\pi A}\int_0^\pi k_2 A \sin\phi \sin\phi \, d\phi = k_2$$

(b) The describing function analysis says that the Nyquist curve $G(i\omega)$ is not allowed to encircle or intersect $-1/N(A)$, which lies in the interval $[-1/k_1, -/k_2]$, while the circle criterion postulates that $G(i\omega)$ is not allowed to encircle the disc going through the point $-1/k_1$ and $-1/k_2$. Hence, stability analysis using describing function analysis is only indicative and does not give sufficient conditions for closed loop stability.

3. (a) Write the system $\dot{x} = f(x)$. Notice that $x = 0$ is an equilibrium of the system since $f(0) = 0$. Linearization around the origin gives

$$A = \frac{d}{dx} f(x)|_{x=0} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

with eigenvalues $0.5 + / - 0.9i$. Hence we have an unstable focus. The origin is locally unstable, which means that trajectories move away from the origin.

(b) Consider the unit circle $\gamma(x) = x_1^2 + x_2^2 - 1 = 0$. We can then show that $\gamma(x)$ describes a limit cycle since,

$$\begin{aligned} \frac{d}{dt}\gamma(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1x_2 + 2x_2(-x_1 - \gamma(x)x_2) = -2x_2^2\gamma(x) \end{aligned}$$

Hence $\gamma(x)$ is an invariant set (in this case a limit cycle). The dynamics on $\gamma(x)$ is given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c) Consider the function $V(x) = (1 - x_1^2 - x_2^2)^2$, which is positive, except when $\gamma(x) = 0$, and radially unbounded. Then

$$\dot{V} = -2x_2^2(x_1^2 + x_2^2 - 1)^2$$

which is zero on $\gamma(x) = 0$ or $x_2 = 0$. If $x_2 = 0$ we get the dynamics $\dot{x}_1 = 0$ and $\dot{x}_2 = -x_1$, which implies $x_1 = 0$. We have already proven that the set $M = \{x|\gamma(x) = 0\}$ is an invariant set. LaSalle's invariant set theorem now gives that all trajectories (not starting at $x = 0$) converge towards M.

4. Notice that $K_P = 1$.

   (a) We see that without saturation

$$U(s) = K_p(E(s) - \left(\frac{1}{F(s)} - \frac{1}{K_p}\right) U(s)), \ \Rightarrow\ U(s) = F(s)E(s)$$

   (b) Notice that $1/F(s) - 1/K_p = -1/(s+1)$ is stable as has static gain 1. Hence, $V = E + 1/(s+1)U$ and the steady state value equals $v = e + u_{max}$.

   (c) With $K_p = 1$, $T_s = 1$

$$V = (E + \frac{1}{s}(E + U - V) \ \Rightarrow\ V = E + \frac{1}{s+1}U.$$

   Hence, the steady state value equals $v = e + u_{max}$.

5. (a) Integration of $\dot{V}(x) \leq \gamma^2 u^2(t) - y^2(t)$ gives

$$\int_0^t y^2(\tau)d\tau \leq \gamma^2 \int_0^t u^2(\tau)d\tau - [V(x(t)) - V(x(0))]$$

   Now $x(0) = 0$ and $V(0) = 0$ and $V(x) \geq 0$ imply that

$$\int_0^t y^2(\tau)d\tau \leq \gamma^2 \int_0^t u^2(\tau)d\tau$$

   Let $t \to \infty$ to prove that a bounded input gives a bounded output.

   (b)

$$\begin{aligned} &\begin{bmatrix} x^T & u \end{bmatrix} \begin{bmatrix} A^TP + PA + C^TC & PB \\ B^TP & -\gamma^2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ &= x^T(A^TP + PA + C^TC)x + 2x^TPBu - \gamma^2 u^2 \leq 0 \end{aligned}$$

   Identify $y = Cx$ to rewrite it as

$$x^T(A^TP + PA)x + 2x^TPBu \leq \gamma^2 u^2 - y^2 = s(u, y)$$

   Take $V(x) = x^TPx \geq 0$,

$$\dot{V} = \dot{x}^TPx + x^TP\dot{x} = x^T(A^TP + PA)x + 2x^TPBu$$

   and hence $\dot{V} \leq s(u, y)$. Hence, the linear system is dissipative.