AUTOMATIC CONTROL
Department of Signals, Sensors & Systems, KTH

Nonlinear Control, 2E1262

Exam 8.00–13.00 March 17, 2005

Aid:

Lecture-notes from the nonlinear control course and textbook from the basic course in control (Glad, Ljung: Reglerteknik, or similar approved text). Mathematical handbook (*e.g.* Beta Mathematics Handbook). Other textbooks, exercises, solutions, calculators, etc. are **not** allowed.

Observandum:

- Name and social security number(*personnummer*) on every page.
- Only one solution per page.
- Do only write on one side per sheet.
- Each answer has to be motivated.
- Specify the total number of handed in pages on the cover.
- The exam consists of five 10 credit problems.

Grading:

Grade 3: ≥ 23
Grade 4: ≥ 33
Grade 5: ≥ 43

Results:

The results will be posted within 2005-03-30 on the department's board, Osquldas väg 10, second floor. **If you want your result emailed, please state this and include your email address.**

Responsible: Bo Wahlberg 790 7242, Alberto Speranzon 790 73 26

Good Luck!

1.

(a) [2p] A scalar nonlinear system is described by the differential equation

$$\dot{x}(t) = f(x(t)), \quad x \in \mathbb{R}$$

where the nonlinear function satisfies

$$\begin{aligned} f(0) &= 0 \\ \frac{d}{dx} f(x)|_{x=0} &< 0 \end{aligned}$$

Show that $x = 0$ is an asymptotically stable equilibrium point.

(b) [3p] Solve the scalar differential equation

$$\dot{x}(t) = -x^2(t), \quad x(0) = 0.01 \quad \text{or} \quad x(0) = -0.01$$

For which of the two initial points is the solution bounded?

(c) Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -h(x_1) + u(t) \end{aligned}$$

where $h(0) = 0$ an $zh(z) > 0$ for all $z \neq 0$. Consider the Lyapunov function candidate

$$V = \int_0^{x_1} h(z)dz + \frac{x_2^2}{2}$$

(i) [2p] Verify that $V(x)$ satisfies $V(0) = 0$, $V(x) > 0$, $x \neq 0$ and calculate $\dot{V}(x)$.

(ii) [3p] Assume the feedback law $u = -\sigma(x_2)$. Use i) to give conditions on the function $\sigma(x_2)$ so that the *closed loop system* is stable at $x = 0$.

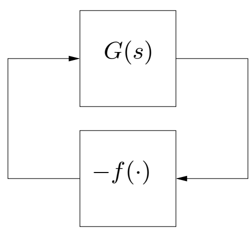
2. Suppose that the static nonlinear function $f(\cdot)$ is odd and satisfies

$$k_1 x^2 \leq x f(x) \leq k_2 x^2, \quad \forall x$$

(a) [7p] Show that the describing function $N(A)$ for $f(\cdot)$ satisfies

$$k_1 \leq N(A) \leq k_2$$

(b) [3p] Consider the feedback system



Here $G(s)$ is a stable linear system and f is the nonlinear function described above. Discuss how describing function analysis of this closed loop system relates to stability analysis using the circle criterion.

3. A plant can be described by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) + [1 - x_1^2(t) - x_2^2(t)]x_2(t)\end{aligned}$$

(a) [3p] Discuss the stability of the origin.

(b) [2p] Find the limit cycle of the system.

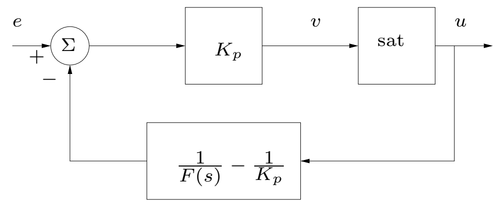
(c) [5p] Prove using LaSalles theorem that all trajectories not starting from the origin converge to the limit cycle.

Hint: Try $V(x) = (1 - x_1^2(t) - x_2^2(t))^2$

4. We will study two ways to implement the PI controller

$$U(s) = F(s)E(s), \quad F(s) = 1 + \frac{1}{s}$$

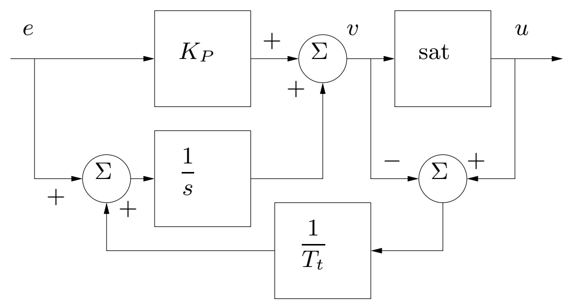
in order to avoid windup for saturated control signals. Let $K_p = \lim_{s\to\infty} F(s) = 1$ and study the implementation



(a) [2p] Show that the transfer function from e to u equals $F(s)$ in case of no saturation.

(b) [3p] What is the steady state value of the output v from the regulator in case of saturation $u = u_{max}$ and a constant e?

(c) [5p]] Consider anti-windup based on tracking as described in Lecture 7. Take $T_t = 1$ (i.e. equal to the integration time of the PI regulator). We then get the following block diagram



What is then the steady state value of the output v from the regulator in case of saturation $u = u_{max}$ and a constant e?

5. A nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x,u), \quad x(0)=0 \\ y(t) &= h(x,u) \end{aligned}$$

is called dissipative if there exists a storage function $V(x)$ such that

$$\begin{aligned} V(0) &= 0, \quad V(x) \geq 0, \ \forall x \neq 0 \\ \dot{V}(x) &\leq s(u,y), \quad \text{along trajectories } x(t) \text{ with } x(0)=0. \end{aligned}$$

The function $s(u,y)$ is called the supply function. A common choice, treated in the course, is $s(u,y) = uy$, which lead to the concept of passivity. Another common choice is

$$s(u,y) = \gamma^2 u^2(t) - y^2(t), \quad \gamma > 0$$

(a) [5p] Show that if a system is dissipative with respect to the supply rate $s(u,y) = \gamma^2 u^2(t) - y^2(t)$, $\gamma > 0$, then the system is bounded-input bounded-output stable.

(b) [5p] Consider a single input single output linear system given by the state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0)=0 \\ y(t) &= Cx(t) \end{aligned}$$

Assume that there exist matrix $P \geq 0$ that satisfies the matrix inequality

$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 \end{bmatrix} \leq 0$$

(the big matrix is negative semi-definite)

Show that this implies that the linear system is dissipative with respect to the supply rate $s(u,y) = \gamma^2 u^2(t) - y^2(t)$, and hence bounded-input bounded-output stable. This result is part of the bounded real lemma.

Hint: Try $V(x) = x^T P x$.