Solutions to the exam in Nonlinear Control
2E1262, 2005-12-15

1. (a) (i)-D, (ii)-C, (iii)-A, (iv)-B
   (b) $\dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2) < 0$ if $0 < x_1^2 + x_2^2 < 2$. Hence, the origin is a locally stable.
   (c) The principle of superposition from linear system theory does not hold for general nonlinear systems. Nonlinear may have several distinct stationary point. Nonlinear system can be locally stable, but not globally stable. Nonlinear systems can have finite escape time.

2. (a) Rewriting the system as

$$\dot{x}_1 = -x_1 - x_2 - u$$
$$\dot{x}_2 = x_1$$
$$u = -sgn(\sigma) \qquad \sigma = -x_1 - 2x_2$$

   The switch curve is $\sigma = -x_1 - 2x_2 = 0$. Calculation of u_{eq} gives the sliding set.

$$\sigma = -x_1 - 2x_2 = 0 \rightarrow x_1 = -2x_2$$

$$\dot{\sigma} = -\dot{x}_1 - 2\dot{x}_2 = -x_1 + x_2 + u_{eq} = 0 \rightarrow u_{eq} = x_1 - x_2 = -3x_2$$

   $u_{eq} \in [-1, 1]$ so can only satisfy $u_{eq} = -3x_2$ on the interval $\{x_1 = -2x_2, x_2 \in [-1/3, 1/3]\}$
   The sliding dynamics with the calculated u_{eq} inserted in the system

$$\dot{x}_1 = -x_1 - x_2 - (x_1 - x_2) = -2x_1$$
$$\dot{x}_2 = x_1 = -2x_2$$

   Thus the system is stable along the sliding set.
   (b) Start with the system $\dot{x}_1 = x_1^2 + \phi(x_1)$ which can be stabilized using $\phi(x_1) = -x_1^2 - x_1$. Notice that $\phi(0) = 0$. Take $V_1(x_1) = x_1^2/2$. To backstep, define $z_2 = (x_2 - \phi(x_1)) = x_2 + x_1^2 + x_1$, to transfer the system into the form

$$\dot{x}_1 = -x_1 + z_2$$
$$\dot{z}_2 = u + (1 + 2x_1)(-x_1 + z_2)$$

   Taking $V = V_1(x_1) + z_2^2/2$ as a Lyapunov function gives

$$\dot{V} = x_1(-x_1 + z_2) + z_2(u + (1 + 2x_1)(-x_1 + z_2)) = -x_1^2 - z_2^2$$

   if

$$u = -2x_1 - 2x_2 - 2x_1(x_1 + x_1^2 + x_2)$$

   Hence, the origin is globally asymptotically stable.

3. (a)

$$\dddot{z} + \ddot{z} + \dot{z} = -\frac{1}{3}z^3 = u.$$

Take the Laplace transform on both sides

$$s^3 Z + s^2 Z + sZ = U$$

$$Z = \frac{1}{s(s^2+s+1)}U$$

Let $y = z$ then $u = -\frac{1}{3}y^3$ and $Y = Z$ which gives

$$G(s) = \frac{1}{s(s^2+s+1)}$$

(b) The function is odd $\Rightarrow a_0 = a_1 = 0$.

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(A\sin(\phi))\sin(\phi)d\phi = \frac{A^3}{3\pi}\int_0^{2\pi}\sin(\phi)^4 d\phi = \frac{A^3}{4}$$

$$N(A) = \frac{ia_1 + b_1}{A} = \frac{A^2}{4}$$

(c) Possible limit cycles occur when

$$G(iw) = -\frac{1}{N(A)}.$$

Because $N(A)$ is real we want to calculate the points were $Im(G(i\omega)) = 0$.

$$G(i\omega) = \frac{-i}{\omega(-\omega^2 + i\omega + 1)} = \frac{-i(1-\omega^2 - i\omega)}{\omega((1-\omega^2)^2 + \omega^2)}$$

so

$$ImG(i\omega) = \frac{-(1-\omega^2)}{\omega((1-\omega^2)^2+\omega^2)} = 0$$

which gives $w = \pm 1$. The only valid solution is thus $w = 1$.

$$G(i1) = -1 = -\frac{1}{N(A)} = -\frac{4}{A^2} \Rightarrow A = \pm 2$$

So finally $\omega = 1$ and $A = 2$.

4. (a) Using the proposed $V(x) = x^T P x$ and the linearization

$$\dot{x}_1 = -x_1 + 2x_2$$
$$\dot{x}_2 = -2x_1$$

give

$$\dot{V} = x^T(AP + PA^T)x = -x_1^2 - x_2^2 = -||x||^2$$

The matrix P has eigenvalues 0.8 and 1.3 and is thus positive definite. Hence, the linear system is asymptotically stable

(b) Let $g(x) = (x_1 x_2 \;\; -x_1^2)^T$ Then $||g||^2 = x_1^2 x_2^2 + x_1^4 = |x_1|^2 ||x||^2$. Hence, $||g|| \leq \gamma ||x||$ for $x \in D$. We now have a similar setup as in the proof of Lyapunov's Linearization Method in Lecture 4, i.e.

$$\begin{aligned}\dot{V} &= -||x||^2 + 2x^T P g(x) \leq -||x||^2 + 2||x^T P g(x)|| \\ &\leq -||x||^2 + 2\gamma\lambda_{max}(P)||x||^2 = -(1 - 2\gamma\lambda_{max}(P))||x||^2\end{aligned}$$

Hence the the nonlinear system is locally asymptotically stable if $\gamma < 1/(2\lambda_{max}(P)) = 0.37$.

5. (a) Setting $V(x) = 0.5x^2$ and using the state equation give

$$uy = (\dot{x} + x)x = \dot{V} + x^2 = \dot{V} + y^2$$

Hence, we can take $\delta = 1$.

(b)

$$\dot{V} \leq uy - \delta y^2 = -\frac{1}{2\delta}(u - \delta y)^2 + \frac{1}{2\delta}u^2 - \frac{\delta}{2}y^2 \leq \frac{1}{2\delta}u^2 - \frac{\delta}{2}y^2$$

Integrating both sides over $[0, T]$ gives

$$\int_0^T y^2(\tau)d\tau \leq \frac{1}{\delta^2}\int_0^T u^2(\tau)d\tau - \frac{2}{\delta}(V(x(T)) - V(x(0))) \leq \frac{1}{\delta^2}\int_0^T u^2(\tau)d\tau$$

where we have used that $V \geq 0$ and $V(x(0)) = 0$. By letting $T \to \infty$ we show BIBO stability.