Proposed solutions to the exam in Nonlinear Control
2E1262, 2006-12-18

1. (a) The equilibria x^* is given by

$$\dot{x} = \begin{pmatrix} -x_2 \\ -x_2 + \sin(x_1) - \text{sat}(x_2) \end{pmatrix} = 0,$$

so we have $(x_1^*, x_2^*) = (n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$. Linearizing around the nth equilibria gives

$$A_n = \begin{pmatrix} 0 & -1 \\ \cos(x_1^*) & -2 \end{pmatrix}\Bigg|_{x_1^* = n\pi} = \begin{pmatrix} 0 & -1 \\ (-1)^n & -2 \end{pmatrix}.$$

Calculating the eigenvalues of A_n gives

$$\det(\lambda_n I - A_n) = \lambda_n^2 + 2\lambda_n + (-1)^n = 0 \quad \Rightarrow$$
$$\lambda_n = \begin{cases} (-1, -1), & \text{if } n \text{ even}, \\ (-1 - \sqrt{2}, -1 + \sqrt{2}), & \text{if } n \text{ odd}, \end{cases}$$

with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 + \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 - \sqrt{2} \end{pmatrix}.$$

We conclude that each "even equilibrium" is a *stable* node converging along the $(1, 1)^T$ direction while each "odd equilibrium" is an saddle point, and hence *unstable*, where we will approach the equilibrium point along the $(1, 1+\sqrt{2})^T$ direction and escape along the $(1, 1-\sqrt{2})^T$. The phase plot for the two different types of equilibriums are displayed in Figure 1.

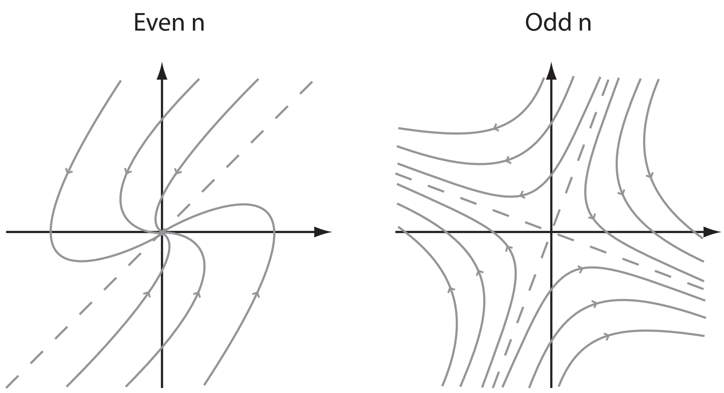

Figure 1: Phase plot sketch for the two different types of equilibrium points.

(b) (i) We need to consider four different cases:

1. For $x < -5$, we have $\dot{x} < 0$, so x will decrease towards $-\infty$.
2. For $-5 < x < 5$, we have $\dot{x} > 0$, so x will increase towards $x_0 = 5$.
3. For $5 < x < 10$, we have $\dot{x} < 0$, so x will decrease towards $x_0 = 5$.
4. For $x > 10$, we have $\dot{x} > 0$, so x will increase towards $+\infty$.

Thus x stays bounded for $-5 \leq x(0) \leq 10$.

(ii) Equilibrium points are given by $f(x) = 0$, from Figure 1 we get $x_0 = \{-5, 5, 10\}$.

(iii) Studying small perturbations δx around the equilibrium x_0 we have

$$\begin{aligned}\dot{x} &= f(x) - K(x - x_0) = f(x_0 + \delta x) - K\delta x \\ &= f(x_0) + f'(x_0)\delta x + \mathcal{O}(\delta x^2) - K\delta x \approx (f'(x_0) - K)\delta x.\end{aligned}$$

For local stability we need that $\dot{x} < 0$ when $x > x_0$, and $\dot{x} > 0$ when $x < x_0$. To fulfill this we need the slope K of the linear control to be greater than the slope of the nonlinearity. From Figure 2 we get an approximate slope of ≈ 150 at $x_0 = -5$ and ≈ 80 at $x_0 = 10$. To dominate this with the linear control we thus need $K > 150$ and $K > 80$ respectively.

2. (a) We have that

$$\dot{x} = Ax, \quad A = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}. \tag{1}$$

Since A triangular it is to see that $\text{eig}(A) = \{-1, -1\}$. Hence the system is stable (i.e A is Hurwitz).

(b) $V(0) = 0$, $V(x) = x^T Q x = x^T \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} x > 0$ for all $x \neq 0$, since Q is positive definite. Furthermore, since $V(x)$ is a quadratic form we can conclude that $V(x)$ is radially unbounded. It remains to check if $\dot{V}(x) < 0$,

$$\dot{V}(x) = -2(x_1^2 + x_2^2) < 0 \text{ for all } x \neq 0 \text{ and } \lambda > 0.$$

By invoking Lyapunov's theorem for global asymptotic stability we can conclude that the system is globally asymptotically stable.

(c) $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, $V(x)$ radially unbounded, and $\dot{V}(x) = 2x_1\dot{x_1} + 2x_2\dot{x_2} = -2(x_1 + x_2)^2 \leq 0$ Lyapunov's direct method now only gives local stability, since $\dot{V} = 0$ if $x_1 + x_2 = 0$. However, since $\dot{x_1} + \dot{x_2} = -3x_1 - x_2 \neq 0$ if $x_1 + x_2 = 0$, LaSalles Theorem for Global Asymptotically Stability can be applied to prove the same result as in b).

(d) We have that $x_2 = \lambda = e^{-t}$, so $\dot{x}_2 = -e^{-t} = -x_2$. Subsequently the extended state space becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 x_2 \\ -x_2 \end{pmatrix}, \tag{2}$$

which is nonlinear due to the $x_1 x_2$. The subsystem for x_1 is however linear time varying.

3. (a) Consider the Lyapunov function *candidate* $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. We have that

$$\begin{aligned} V(0,0) &= 0, \\ V(x_1, x_2) &> 0 \text{ when } (x_1, x_2) \neq (0,0), \\ V(x_1, x_2) &\to \infty \text{ when } \|x\| \to \infty. \end{aligned}$$

We need $\dot{V}(x_1, x_2) < 0$ for global asymptotic stability:

$$\dot{V}(x_1, x_2) = x_1\dot{x_1} + x_2\dot{x_2} = x_2(x_1^2 + x_1^5 + u).$$

By choosing

$$u = -x_1^5 - x_1^2 - x^2$$

we get

$$\dot{V}(x_1, x_2) = -x_2^2 \leq 0$$

which implies stability, but *not* asymptotic stability. Obviously $\dot{V} = 0$ along the line $x_2 = 0$. However we have that

$$\dot{x}_2 = x_1^5 + u = x_1^5 - x_1^5 - x_1^2 - x_2 = -x_1^2$$

along this line. Thus $\dot{x}_2 \neq 0$ whenever $x_1 \neq 0$, and we can hence conclude that a solution trajectory cannot stay on the x_1–axis except at the origin. This means that the only *invariant subset* of the line $x_2 = 0$ is the origin, and hence we have global asymptotic stability for the origin.

4. (a) The switching curve is given by $\tilde{S} = \{x \in \mathbb{R}^2 \mid \sigma(x) = 0\}$ where $u = -\text{sign}(\sigma(x))$, so $\sigma(x) = x_1 + x_2$ in our case. Hence the switching curve is a line with slope -1 in $\mathbb{R}^2$. The equivalent control is defined by $u_{\text{eq}} = \{u \in \mathbb{R} \mid -1 \leq u \leq 1, \dot{\sigma}(x) = 0, \sigma(x) = 0\}$. Now

$$\dot{\sigma}(x) = \dot{x}_1 + \dot{x}_2 = -x_2 - 2x_1 + u_{\text{eq}} + x_1 = 0 \quad \Rightarrow$$
$$u_{\text{eq}} = x_1 + x_2 = 0 \text{ on } \tilde{S}.$$

Since $u_{\text{eq}} = 0 \in [-1, 1]$ we have that the hole line $x_1 + x_2 = 0$ belongs to the sliding set S, i.e. $S = \tilde{S}$.
On S we have the sliding dynamics

$$\dot{x}_1 = -x_2 - 2x_1 + u_{\text{eq}} = -x_1,$$
$$\dot{x}_2 = x_1 = -x_2.$$

Thus the system will be asymptotically stable and slide to the origin, see Figure 2.

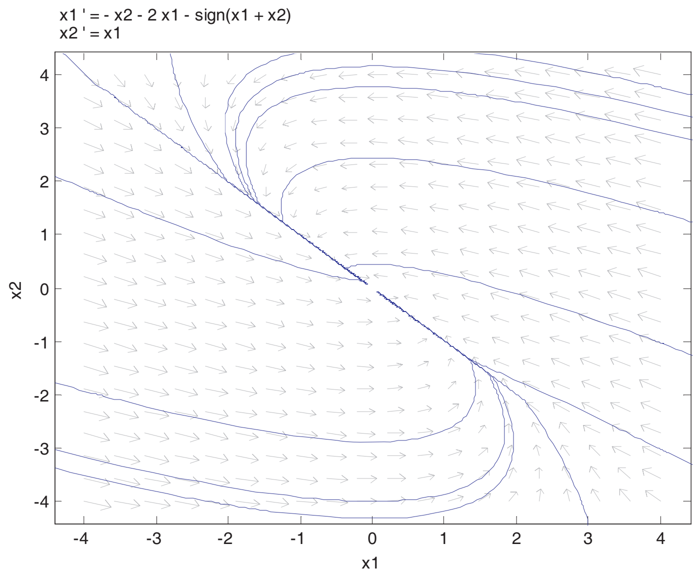

Figure 2: The phase plot in Problem 4.

(b) The two systems have similar structure, a natural first approach is therefore $x_1 = z_1$ and $x_2 = z_2^3$ (compare e.g. the right hand sides of the equations for $\dot{x}_1$ and $\dot{z}_1$ respectively):

$$x_1 = z_1 \Rightarrow \dot{x}_1 = \dot{z}_1 = -z_2^3 - 2z_1 - \text{sign}(z_1 + z_2^3)$$

where direct comparison of terms matches. For the second state equation we have

$$x_2 = z_2^3 \Rightarrow \dot{x}_2 = 3z_2^2\dot{z}_2 \Rightarrow \dot{z}_2 = \frac{\dot{x}_2}{3z_2^2} = \frac{x_1}{3z_2^2} = \frac{z_1}{3z_2^2}.$$

Thus, the proposed change of coordinates converts between the two system descriptions.

5. (a) Set $T = \infty$

$$\langle y, u\rangle = \int_0^\infty y(t)u(t)dt = \{ \text{ Parseval's theorem } \}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(i\omega)U(i\omega)d\omega = \{\, Y(i\omega) = G(i\omega)U(i\omega) \,\}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(i\omega)U(i\omega)G(i\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(i\omega)|^2 G(i\omega)d\omega$$
$$= \{ \text{ the integral must be real } \} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(i\omega)|^2 \mathrm{Re}\,(G(i\omega))\, d\omega$$
$$\geq \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(i\omega)|^2 \epsilon |G(i\omega)|^2 d\omega = \epsilon\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)U(i\omega)|^2 d\omega$$
$$= \epsilon\frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega = \{ \text{ Parseval's theorem } \}$$
$$= \epsilon\int_0^\infty |y(t)|^2 dt = \epsilon|y|^2$$

(b) We have

$$\mathrm{Re}\,(G(i\omega)) = \mathrm{Re}\left(\frac{1}{i\omega+1}\right) = \mathrm{Re}\left(\frac{-i\omega+1}{\omega^2+1}\right)$$
$$= \mathrm{Re}\left(\frac{1}{\omega^2+1} - i\frac{\omega}{\omega^2+1}\right) = \frac{1}{\omega^2+1} = |G(i\omega)|^2.$$

Hence there exist an $\epsilon > 0$ (actually all $\epsilon \in (0,1]$) such that

$$\mathrm{Re}\,(G(i\omega)) \geq \epsilon|G(i\omega)|^2,$$

thus the system is output strictly passive according to problem (5a).

(c)

$$\langle y, u\rangle = \int_0^\infty y(t)u(t)dt = \int_0^\infty f(u(t))u(t)dt \geq \int_0^\infty \epsilon f^2(u(t))dt$$
$$= \epsilon\int_0^\infty |y(t)|^2 dt = \epsilon|y|^2$$

(d) Here we have $xf(x) = (1+|x|)f(x)^2 \geq \epsilon f(x)^2$ for all $\epsilon \in (0,1]$ thus the system is output strictly passive according to 5c).