Proposed solutions to the exam in Nonlinear Control EL2620, 2007-12-17

1. (a) All systems except (ii) have a unique equilibrium at the origin. Thus, (ii) corresponds to A. System (iii) is linear and has complex eigenvalues with real part in the RHP, thus an unstable focus corresponding to B. System (iv) also has a trivial linearization (all parts linear except x_2^3 which has no linear part) and the eigenvalus are $-1, 1$ and hence an unstable saddle corresponding to C. System (i) has the same linearization as (iii), hence an unstable focus but the nonlinearity allows for the limit cycle in D.

   (b)
$$u = x^2 - sin(x) - x + r$$

   (c) (i) Differentiating the output yields
$$\dot{y} = \dot{x}_1 = x_2 - x_1^2$$
   Since u does not appear, we differentiate once again
$$\ddot{y} = \dot{x}_2 - 2x_1\dot{x}_1 = -x_2x_1^2 + u - 2x_1(x_2 - x_1^2)$$
   and setting $\ddot{y} = r$ yields
$$u = x_2x_1^2 + 2x_1(x_2 - x_1^2) + r$$
   (ii) Since the relative degree in this case equals the order of the system, there are no zero dynamics and hence unstable zero dynamics will not be a problem.

2. (a) (i) From the first equation with $\dot{x}_1 = 0$ we get $x_2 = 0$ or $x_1 = 1 + x_2^2$. Inserting this into the second equation with $\dot{x}_2 = 0$ yields $k_2 = 0$, which is not satisfied, and $k_2 - 5x_2 = 0$ or
$$x_2^* = k_2/5 = 2 \ , \quad x_1^* = 1 + x_2^{*2} = 5$$
   Linearization yields
$$A = \frac{1}{1 + x_2^{*2}} \begin{pmatrix} -k_1x_2^* & 2k_1x_2^{*2} \\ -4x_2^* & -5 + 3x_2^{*2} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -6 & 24 \\ -8 & 7 \end{pmatrix}$$
   And the characteristic equation is given by
$$(5\lambda + 6)(5\lambda - 7) + 184 = 0 \Leftrightarrow 25\lambda^2 - 5\lambda + 142 = 0$$
   which has at least one root in the open complex right half plane since the coefficients have different signs. Hence, the equilibrium is unstable according to Lyapunov's linearization method.

(ii) The region K is a rectangle in first quadrant of the state plane, and invariance implies that all trajectories point inwards at the boundary of the rectangle.

* At the line $x_1 = 0, x_2 > 0$ we get $\dot{x}_1 = k_1 x_2 > 0$ and hence all trajectories point inwards
* At the line $x_2 = 0, x_1 > 0$ we get $\dot{x}_2 = k_2 > 0$ and hence all trajectories point inwards
* At the line $x_1 = 1+k_2^2, x_2 > 0$ we get $\dot{x}_1 = k_1 x_2 \left(1 - \frac{1+k_2^2}{1+x_2^2}\right) < 0$ since $0 < x_2 < k_2$ and hence the trajectories point inwards.
* At the line $x_2 = k_2$ we get $\dot{x}_2 = -\frac{4x_1 k_2}{1+k_2^2} < 0$ since $x_1 > 0$ and $k_2 > 0$ and hence all trajectories point inwards

Thus, we have shown that all trajectories at the boundary of K point inwards and hence K is invariant.

(iii) Since the only equilibrium is unstable and within K, there must exist some other stable behavior within K. This must be a limit cycle since the system is two-dimensional[1].

(b) (i) The Jacobian at $x = (0, 0)$ has eigenvalues $\lambda_{1,2} = \pm i$ and since they are on the imaginary axis we can not deduce anything about the local stability from Lyapunov's linearization method.

(ii) Trivially, $V(x) > 0$ and $V \to \infty$ as $\|x\| \to \infty$.

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1 x_2 + x_2(-x_1 - x_2^3) = -x_2^4$$

Hence, $\dot{V} \leq 0 \quad \forall x$. Now, $\dot{V} = 0$ on the line $x_2 = 0$. However, with $x_2 = 0$ we get $\dot{x}_2 = -x_1$ which is zero only if $x_1 = 0$. Hence, the equilibrium at the origin is the only invariant set on $x_2 = 0$ and hence LaSalle theorem gives that the equilibrium is globally asymptotically stable.

3. (a) Trivially $V > 0$ and $V \to \infty$ as $\|x\| \to \infty$.

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^2 x_2 + x_1 u + x_2 x_1^2 - x_2^2 = x_1 u - x_2^2$$

which is made strictly negative for all $x \neq 0$ if $u = -kx_1$ with $k > 0$.

(b) The system is on strict feedback form. Starting by stabilizing the first equation using $x_2 = \phi(x_1) = x_1^2 - x_1$, we get that $V(x_1) = 0.5x_1^2$ is a Lyapunov function. The stabilizing control is then given by

$$u = (2x_1 - 1)(-x_1^2 + x_2) - x_1 - (x_2 - x_1^2 + x_1) + x_2 - \frac{1}{1 + x_1^2}$$

[1] For higher dimensional systems there exist other possible non-equilibrium stable behaviors; quasi-periodic and chaotic solutions, but that is out of the scope of this course.

or

$$u = -2x_1(x_1^2 - x_1 - x_2 + 1) - x_2 - \frac{1}{1 + x_1^2}$$

with corresponding Lyapunov function

$$V(x_1, x_2) = 0.5x_1^2 + (x_2 - x_1^2 + x_1)^2$$

4. The system corresponds to a feedback loop with a static nonlinearity $f(u)$ and a stable linear system $KG(s)$.

   (i) The small gain states that the loop is stable if the loop-gain $\gamma < 1$. The nonlinearity has $f(u)/u = 1/(1 + u^2)$ which has a maximum $\gamma_f = 1$ (at $u = 0$). The linear system has gain $\gamma_G = \sup_\omega |KG(i\omega)| = |K|$. Hence, the loop-gain $\gamma < 1$ if $|K| < 1$, or $-1 < K < 1$, which then guarantees closed-loop BIBO stability.

   (ii) The sector of the nonlinearity has $k_1 = 0$ and $k_2 = 1$ and hence the circle criterion yields closed-loop stability if $ReKG(i\omega) > -1$. We have $ReG(i\omega) = (1 - \omega^2)/((1 - \omega^2)^2 + 4\omega^2)$ which has a minimum $ReG = -0.125$ which yields $K < 8$. The maximum of $ReG(i\omega) = 1$ which yields $K > -1$. Thus, the closed-loop is guaranteed stable if $-1 < K < 8$.

   (iii) The static nonlinearity is odd, i.e., $f(-u) = -f(u)$ and hence the describing function $N(A)$ is real. Since $f(u)/u > 0$, $N(A) > 0$. Thus, $-1/N(A)$ is located on the negative real axis in the complex plane. Since the linear system frequency response $KG(i\omega)$ never crosses the negative real axis for $K > 0$, the describing function method do to predict oscillations for any $K > 0$.

   (iv) The methods in (i) and (ii) are both based on the small gain theorem and hence conservative, that is, sufficient but not necessary. The circle criterion is in general less conservative than the direct application of the SGT to the loop, as can also be seen by the prediction. The describing function method do not provide neither sufficient nor neccessary conditions for the existence or absence of an oscillation. However, the result may be taken as an indication that the system is stable for all $K > 0$. However, this needs to be verified by some more rigorous method, e.g., using Lyapunov theory.

5. (a) Trivially $V(y) > 0$ and

   $$\dot{V} = y\dot{y} = y(-x_1 + 2x_2 - x_1 + x_2 - u) = y(-2x_1 + 3x_2 - u)$$

   To satisfy $\dot{V} < 0$ we employ the control

   $$u = -2x_1 + 3x_2 + \beta sign(y), \quad \beta > 0$$

With this controller $\dot{y} = -\beta sign(y)$ which hence brings the output to zero in finite time $t_s = |y(0)|/\beta$.

(b) We have $y(0) = a - b$ and if $a - b > 0$ we get $dy/dt = -\beta$, i.e., $y(t)$ is a straight line with slope $-\beta$, untill $y = 0$ at $t_s = (a - b)/\beta$. If $a - b < 0$, $dy/dt = \beta$ untill $y = 0$ at $t_s = (b - a)/\beta$. The sketch is not shown here.

(c) At the sliding mode $y = 0$ we require $\dot{y} = 0$ and hence

$$-x_1 + 2x_2 - x_1 + x_2 - u_{eq} = 0 \quad \Rightarrow \quad u_{eq} = -2x_1 + 3x_2$$

With $u = u_{eq}$ we get

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2 \\ \dot{x}_2 &= -x_1 + 2x_2 \\ y &= x_1 - x_2 \end{aligned}$$

and the observability matrix becomes

$$O = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

which is rank deficient and hence the system is unobservable. The unobservable subspace equals $\ker O = [1\ 1]^T$, i.e., $x_1 = x_2$ which corresponds to $y = x_1 - x_2 = 0$, i.e., the states at the sliding mode. The sliding mode dynamics correspond to $x_1 = x_2$ and hence $\dot{x}_1 = -x_1 + 2x_1 = x_1$ which is unstable. The same applies to x_2. Thus, the dynamics on the sliding mode are unobservable and unstable, which is unacceptable since it implies that the states and the control input grows exponentially while $y = 0$.

(d) The fact that the system is unstable when we keep $y = 0$ implies that the system has unstable zero dynamics, or since this is a linear system, zeros in the complex right half plane. This again implies that it is not possible to force y to 0 without also forcing both states to 0, and this can this can only be achieved in exponential time for a linear system with a single control input. This means that not even a time-optimal controller can take the output to zero in finite time for this system.