Proposed solutions to the exam in Nonlinear Control
EL2620, 2008-05-30

1. (a) All systems except (iv) have a unique equilibrium at the origin. Thus, (iv) corresponds to D. System (i) has a Jacobain at the origin with eigenvalues 2 and -1 and hence corresponds to an unstable saddle; B. System (ii) has 2 eigenvalues at 1, while (iii) has eigenvalues $0.5 \pm \sqrt{3}/2$. System (iii) is hence an unstable focus at the origin, corresponding to C, while (ii) then corresponds to A

   (b)
$$u = x^3 - \frac{1}{1+x^4} - x + r$$

   (c) (i) Differentiating the output yields
$$\dot{y} = \dot{x}_1 = 2x_2 + x_1^3 + u$$
   Since u appears in $\dot{y}$ we can now choose
$$u = -2x_2 - x_1^3 + v$$
   to obtain $\dot{y} = v$.

   (ii) For the closed loop system we obtain
$$\dot{x}_1 = -K_p x_1$$
$$\dot{x}_2 = -x_2 + 2_x 2 + x_1^3 - K_p x_1 = x_2 - K_p x_1 + x_1^3$$
   The corresponding Jacobian at the origin will have eigenvalues $-K_p$ and 1 and will hence always be unstable. The reason is that the zero dynamics are unstable, in this case reflected by x_2 being unstable when $y = x_1 = 0$.

2. (a) The equilibria are $(0,0)$ and $(1,1)$. The Jacobians have eigenvalues $\lambda_1 = 0, \lambda_2 = 0$ and $\lambda_1 = -0.55, \lambda_2 = -5.5$, respectively. The latter solution is thus stable, while conclusion about the stability origin can not be reached from the linearization since the eigenvalues are in the closed left half plane with some on the imaginary axis. There are several ways to show that the origin in fact is unstable, but these have not been covered in the course.

   (b) A sketch of the region traced out by Γ in the state plane shows a banana shape. For each point on the left boundary of Γ $x_2 = x_1^2$ and thus
$$\dot{x}_1 = -x_1^3 + x_1^2 > 0$$
$$\dot{x}_2 = x_1^6 - x_2^3 = 0$$

so the trajectory points inward Γ at every point at the left boundary. At the right boundary $x_2 = x_1^3$, and

$$\dot{x}_1 = 0$$

$$\dot{x}_2 = x_1^6 - x_1^9 > 0$$

so the trajectory points inwards. Hence Γ is invariant.

(c) The equlibirum point is at the boundary of Γ. Draw a trajectory illustrating how a trajectory starting close to the origin moves to $(1, 1)$.

3. With $V = 0.5(x_1^2 + x_2^2)$ we have $V(0,0) = 0$, $V > 0$ when $x \neq 0$, and $\|V\| \to \infty$ when $\|x\| \to \infty$. It remains to show that $\dot{V} < 0$ for all $x \neq 0$.

$$\dot{V} = x_1^3 + x_1^2 x_2 + x_1 u + x_1 x_2^3 + x_1 x_2 = x_1(x_1^2 + x_1 x_2 + u + x_2^3 + x_2)$$

By choosing

$$u = -x_1^2 - x_1 x_2 - x_2^3 - x_2 - x_1$$

we get

$$\dot{V} = -x_1^2$$

This is a strictly negative function for all x except for at the line $x_1 = 0$, which includes the origin. It remains to be shown that the equilibrium which is the only invariant set at the line where $\dot{V} = 0$. For $x_1 = 0$ we get

$$\dot{x}_1 = -x_2^3 - x_2$$

which is zero only when $x_2 = 0$. Thus, the origin is the only invariant set for which the $\dot{V} = 0$ and hence the origin is globally asymptotically stable.

4. (a) The describing function is the amplitude dependent gain, and real valued in this case (why?). The gain will in this case have a maximum $N(A) = 2$ for small input amplitudes A and then drop off as $A > 2$, with further drops for $A > 4$ and $A > 6$. For stability analysis it is usually the maximum amplification that matters (for real valued functions).

(b) The linear system G must be chosen such that the Nyquist curve intersects $-1/N(A)$. In this case $-1/N(A)$ covers the negative real axis up to the point -0.5. Thus, the Nyquist curve of G should cross the negative real axis to the left of -0.5. A possible candidate is

$$G(s) = \frac{4}{(s+1)^3}$$

(c) The gains are $\gamma(f) = 2$, $\gamma(f(f)) = 4$, $\gamma(f + f) = 4$

(d) The system is globally stable with Lyapnuov function $V(x) = x^2$ for all $K > 0$. For example, note that $\dot{V} = -2xf(x) = -2|xf(x)| \leq 0$, for all $x \neq 0$, and that V is radially bounded.

5. (a) We seek a sliding mode controller using the Lyapunov function $V(x) = 0.5x_2^2$. Differentiation yields

$$\dot{V} = x_2\dot{x}_2 = -2x_2^2 + x_1x_2 - 0.5ux_2$$

By choosing

$$u = -4x_2 + 2x_1 + 2sign(x_2)$$

we get $\dot{V} = -x_2 sign(x_2)$ which is strictly negative for all $x_2 \neq 0$. Furthermore, with this controller

$$\dot{x}_2 = -sign(x_2)$$

which will take x_2 to zero in finite time.

(b) Consider the dynamics of the state x_1 on the manifold S.

$$\dot{x}_1 = -x_1 + x_1x_2 - 4x_2 + 2x_1 + 2sign(x_2)$$

and with $x_2 = 0$

$$\dot{x}_1 = x_1$$

which is exponentially unstable with eigenvalue 1. Thus, both x_1 and u will blow up when forcing $x_2 = 0$.

By letting $x_2 = 0$ and $\dot{x}_2 = 0$ in the state space equations we get

$$\dot{x}_1 = -x_1 + 2x_1 = x_1$$

which thus always results when forcing the system to stay on the manifold S, independent of which controller is employed for the task. The reason is that the system has unstable zero dynamics when the output is $y = x_2$.