AUTOMATIC CONTROL
KTH

Nonlinear Control, EL2620 / 2E1262

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1. Consider the nonlinear system

$$\dot{x}_1 = kx_1 - x_2$$
$$\dot{x}_2 = x_1 - x_2^3 + u$$

(a) The equilibriums are given by:

$$x_1 - x_2 = 0$$
$$x_1 - x_2^3 = 0,$$

which gives: $(0,0), (1,1), (-1,-1)$. The Jacobian is

$$A(x_1, x_2) = \begin{pmatrix} 1 & -1 \\ 1 & -3x_2^2 \end{pmatrix}.$$

Insertion of the origin and calculation of the eigenvalues gives

$$\det(\lambda I - A(0,0)) = \begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0$$
$$\Rightarrow \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1^2 - 4} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

The real part is not zero, so the equilibrium is hyperbolic and the linear analysis is valid locally. Complex eigenvalues with positive real part implies that the origin is an unstable focus. Insertion of $(1,1)$ and calculation of the eigenvalues gives

$$\det(\lambda I - A(1,1)) = \begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda + 3 \end{vmatrix} = \lambda^2 + 2\lambda - 2 = 0$$
$$\Rightarrow \lambda = \frac{-2}{2} \pm \frac{1}{2}\sqrt{2^2 + 8} = -1 \pm \sqrt{3}$$

One positive real and one negative real eigenvalue implies that the equilibrium is a saddle point. Insertion of $(-1,-1)$ gives the same Jacobian so it is also a saddle point.

(b) The general linear state feedback $u = ax_1 + bx_2$ yields $u = 0$ for $x = (0,0)$ and hence the origin is still an equilibrium. We are only asked to make the origin locally asymptotically stable so we can linearize the system at the origin and select a, b such that the real part of the eigenvalues is smaller than zero. The Jacobian at the origin is

$$A(0,0) = \begin{pmatrix} 1 & -1 \\ 1 + a & b \end{pmatrix}.$$

The eigenvalues are given by

$$\det(\lambda I - A(0,0)) = \begin{vmatrix} \lambda - 1 & 1 \\ -1 - a & \lambda - b \end{vmatrix} = \lambda^2 - (1+b)\lambda b + a + 1 = 0$$
$$\Rightarrow \lambda = \frac{1+b}{2} \pm \frac{1}{2}\sqrt{(1+b)^2 - 4(1+a+b)}.$$

So we need to select $b < -1$, $a > -1 - b$, for example $a = 2, b = -2$ gives a stable focus.

(c) The Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is continuously differentiable for $x \in \mathbb{R}^2$, positive semidefinite and zero only at the origin, as well as radially unbounded. Its time derivative

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_2^4 < 0 \; \forall x_2 \neq 0$$

is unfortunately zero on the whole x_1-axis, so Lyapunovs theorem for global asymptotic stability is not fulfilled. Let us use LaSalle's invariant set theorem. We can use the previous Lyapunov function to define a compact positively invariant set $\Omega_c = \{(x_1, x_2) \in \mathbb{R}^2 | \, x_1^2 + x_2^2 \leq c\}$. This set is positively invariant since $\dot{V}(x) \leq 0 \; \forall x$, i.e. the state change is directed inwards or along the level curve for any $c \in \mathbb{R}, c \geq 0$. We select the same Lyapunov function and get the set $E = \{(x_1, x_2) \in \Omega_c | \, x_2 = 0\}$. On the x_1-axis the system equations reduces to

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= x_1 \neq 0 \; \forall x_1 \neq 0, \end{aligned}$$

hence the origin constitutes the largest invariant set, $M = (0, 0)$. LaSalle's theorem now guarantees that every solution $x(0) \in \Omega_c$ approaches the origin as $t \to \infty$. It is hence globally asymptotically stable.

2. (a) Consider the dynamic system

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + y^3 - y \end{aligned}$$

(i) We can for every $c \in \mathbb{R}$ view $x^2 + y^2 = c$, which defines the closure of the set E, as a level curve of the Lyapunov function $V(x, y) = x^2 + y^2$. The time derivative of the Lyapunov function is

$$\dot{V}(x, y) = 2x\dot{x} + 2y\dot{y} = 2y^4 - 2y^2.$$

This is negative semidefinite if $2y^2(y^2 - 1) \leq 0$, i.e. if $y^2 \leq 1$. This holds for all points on every level curve with $c \leq 1$, hence the set E is (positively) invariant for all $c \leq 1$.
An alternative point of view, which gives exactly the same result, is to define $\gamma := x^2 + y^2 - c = 0$ as the closure of the set. Calculate the gradient of the curve:

$$\nabla\gamma = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

and make sure it points outwards from the set, so that it is the normal of the closure. Then we project the change of the trajectory on the closure on the normal and check that it is smaller or equal to zero, i.e. that the change points inwards:

$$\dot{\gamma} = \nabla\gamma \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 2x\dot{x} + 2y\dot{y} = 2y^4 - 2y^2.$$

(ii) What can you conclude from the result in (i)? The set $E, c \leq 1$ constitute a region of attraction[1] of some invariant set existing within E.Since $\dot{V}(x, y) < 0$ for all $c \leq 1$, $y \neq 0$, every level curve will define a region of attraction, so the invariant set must lie on the x-axis. If we also determine the equilibrium points of the system then we see that the origin is the only equilibrium on the x-axis and hence the largest invariant set in E. So every solution $x(0)$ starting in the set $E, c \leq 1$ approaches the origin as $t \to \infty$.

(b) A nonlinear system is described by

$$\begin{aligned} \dot{x}_1 &= 2x_1 - \frac{x_2}{1 + x_2^2} + u \\ \dot{x}_2 &= \frac{x_1}{1 + x_1^2} \end{aligned}$$

[1] Not necessarily the whole region of attraction though.

The Lyapunov function $V(x) = x_1^2 + x_2^2$ is continuously differentiable for $x \in \mathbb{R}^2$, positive definite and only zero at the origin, as well as radially unbounded. So if we can choose u such that its time derivative is negative except at the origin, then Lyapunovs theorem for global asymptotic stability is fulfilled. The time derivative is

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 4x_1^2 - \frac{2x_1x_2}{1+x_2^2} + 2x_1u + \frac{2x_1x_2}{1+x_1^2}$$

so let us select

$$u = -2.5x_1 + \frac{x_2}{1+x_2^2} - \frac{x_2}{1+x_1^2}$$

then we have

$$\dot{V}(x) = -x_1^2.$$

Unfortunately $\dot{V}(x) = 0$ on the whole x_2-axis, so Lyapunovs theorem for global asymptotic stability is not fulfilled. Let us instead use LaSalle's theorem for global asymptotic stability, i.e. we additionally need to show that the origin is the only invariant set on the x_2-axis. Alternatively we could do exactly like in 1c, since this theorem a special case of LaSalle's invariant set theorem. On the x_2-axis the system equations reduces to

$$\begin{aligned}\dot{x}_1 &= -x_2 \neq 0 \ \forall x_2 \neq 0\\ \dot{x}_2 &= 0,\end{aligned}$$

hence the origin constitutes the only invariant set on it. LaSalle's theorem now guarantees global asymptotic stability.

3. (a) We shall consider linearizing control of the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + x_2^2 + u\\ \dot{x}_2 &= x_2 - x_1^2\\ y &= x_2\end{aligned}$$

(i) A standard choice for transformation is $z_1 = x_i, z_2 = \dot{x}_i$. To avoid u being included in the definition of our new state variables, we try $z_1 = x_2$, $z_2 = \dot{x}_2 = x_2 - x_1^2$. Then $x_1 = \pm\sqrt{z_1 - z_2}$, $x_2 = z_1$. The system can then be written as

$$\begin{aligned}\dot{z}_1 &= \dot{x}_2 = z_2\\ \dot{z}_2 &= \dot{x}_2 - 2x_1\dot{x}_1 = x_2 - x_1^2 + 2x_1^2 - 2x_1x_2 - 2x_1x_2^2 - 2x_1u\end{aligned}$$

Now the state feedback

$$u = \frac{x_2 + x_1^2 - 2x_1x_2(1+x_2) + v}{2x_1}$$

will linearize the system by canceling out the nonlinearities giving

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= v\end{aligned}$$

The feedback linearization is only valid if $x_1 \neq 0$. When $x_1 \to 0$ then $u \to \infty$, which cannot be implemented in practice. Thus, we can only linearize either the left or right half state-plane[2].

[2] This may appear of little use since the origin is the equlibrium of the system for $u = 0$. However, nonlinear systems are often operated with a non-zero input, i.e., away from the origin in the state-plane in this case, and hence linearization of only one half plane can be relevant.

(ii) Differntiating the output once yields

$$\dot{y} = \dot{x}_2 = x_2 - x_1^2$$

and hence no appearance of the control input u. Differentiating once again yields

$$\ddot{y} = \dot{x}_2 - 2x_1\dot{x}_1$$

which is equal to $\dot{z}_2$ in problem (i). Hence, the same controller as in (i) yields

$$\ddot{y} = v$$

The same limitations as in (i) apply.
The limitations can also be seen by noting that the relative degree, given by the smallest p such that $L_g L_f^{p-1} h(x) \neq 0$, is not defined in all $\mathbf{R}^2$. In particular,

$$L_g L_f h = \ddot{x_2} = \dot{x}_2 - 2x_1\dot{x}_1 = x_2 + x_1^2 - 2x_1x_2 - 2x_1x_2^2 - 2x_1u$$

which is non-zero except for $x = 0$. Thus, the relative degree is not well defined if we include $x_1 = 0$ in the considered domain.

(iii) For $x_1 \neq 0$ the relative degree is equal to the number of states, and hence no zero dynamics.

(b) A mechanical system with servo dynamics and no damping is described by the model

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + z \\ \dot{z} &= -z + u\end{aligned}$$

This system is on strict feedback form so the backstepping procedure can be used to stabilize the states of the system one by one. Let us start by only considering the first state with $x_2 = \phi_1(x_1)$:

$$\dot{x}_1 = \phi_1(x_1).$$

If we for example select $\phi_1(x_1) = -x_1$, then this system can be shown to be globally asymptotically stable with Lyapunov function $V_1(x_1) = 0.5x_1^2$. This Lyapunov function is clearly continuously differentiable for $x_1 \in \mathbb{R}$, positive definite and only zero at the origin, as well as radially unbounded. The derivative is

$$\dot{V}_1(x_1) = x_1\dot{x}_1 = -x_1^2 < 0 \; \forall x_1 \neq 0,$$

which proves global asymptotic stability.
Let us now do a backstep by doing a change of state variables, i.e. introduce $\xi_1 = x_2 - \phi_1(x_1) = x_1 + x_2$, and study the two state system with $z = \phi_2(x_1, \xi_1)$:

$$\begin{aligned}\dot{x}_1 &= \phi_1(x_1) + x_2 - \phi_1(x_1) = -x_1 + \xi_1 \\ \dot{\xi}_1 &= \dot{x}_2 - \dot{\phi}_1(x_1) = -x_1^3 + \phi_2(x_1, \xi_1) + \dot{x}_1 = -x_1^3 + \phi_2(x_1, \xi_1) - x_1 + \xi_1.\end{aligned}$$

If we for example select $\phi_2(x_1, \xi_1) = x_1^3 - 2\xi_1$, then this system can be shown to be globally asymptotically stable with Lyapunov function $V_2(x_1, \xi_1) = V_1(x_1) + 0.5\xi_1^2 = 0.5x_1^2 + 0.5\xi_1^2$. This Lyapunov function is clearly continuously differentiable for $(x_1 \; \xi_1)^T \in \mathbb{R}^2$, positive definite and only zero at the origin, as well as radially unbounded. The derivative is

$$\dot{V}_2(x_1, \xi_1) = x_1\dot{x}_1 + \xi_1\dot{\xi}_1 = -x_1^2 + x_1\xi_1 - \xi_1 x_1 - \xi_1^2 = -x_1^2 - \xi_1^2 < 0 \; \forall (x_1 \; \xi_1) \neq 0,$$

which proves global asymptotic stability.

Let us do an additional backstep by doing a change of state variables, i.e. introduce $\xi_2 = z - \phi_2(x_1, \xi_1) = 2x_1 + 2x_2 + z - x_1^3$, and study the complete three state system:

$$\begin{aligned}
\dot{x}_1 &= -x_1 + \xi_1 \\
\dot{\xi}_1 &= -x_1^3 - x_1 + \xi_1 + \phi_2(x_1, \xi_1) + z - \phi_2(x_1, \xi_1) = -x_1 - \xi_1 + \xi_2 \\
\dot{\xi}_2 &= \dot{z} - \dot{\phi}_2(x_1, \xi_1) = -z + u - 3x_1^2\dot{x}_1 + 2\dot{\xi}_1 \\
&= -z + u + 3x_1^3 - 3x_1^2\xi_1 - 2x_1 - 2\xi_1 + 2\xi_2.
\end{aligned}$$

This system can be made globally asymptotically stable if we design the controller u based on the Lyapunov function $V_3(x_1, \xi_1, \xi_2) = V_2(x_1, \xi_1) + 0.5\xi_2^2 = 0.5x_1^2 + 0.5\xi_1^2 + 0.5\xi_2^2$. This Lyapunov function is clearly continuously differentiable for $(x_1\ \xi_1\ \xi_2)^T \in \mathbb{R}^3$, positive definite and only zero at the origin, as well as radially unbounded. The derivative is

$$\begin{aligned}
\dot{V}_3(x_1, \xi_1, \xi_2) = x_1\dot{x}_1 + \xi_1\dot{\xi}_1 + \xi_2\dot{\xi}_2 = -x_1^2 + x_1\xi_1 - \xi_1 x_1 - \xi_1^2 + \xi_1\xi_2 \\
+ \xi_2(-z + u + 3x_1^3 - 3x_1^2\xi_1 - 2x_1 - 2\xi_1 + 2\xi_2),
\end{aligned}$$

so if we select the controller

$$u = +z - 3x_1^3 + 3x_1^2\xi_1 + 2x_1 + \xi_1 - 3\xi_2,$$

then

$$\dot{V}_3(x_1, \xi_1, \xi_2) = -x_1^2 - \xi_1^2 - \xi_2^2 < 0\ \forall (x_1\ \xi_1\ \xi_2) \neq 0$$

and the origin of the complete system is globally asymptotically stable. Note that when $x_1 = 0, \xi_1 = 0, \xi_2 = 0$ then it implies that the original state variables also are zero, so the origin of the original state-space is mapped by the state transformations to the origin of the new state-space.

4. (a) Denote the input of Δ with v and the output with z. Then we get the transfer function from z to v as

$$v = G_2(1 - G_1G_3G_2)^{-1}z = Gz,$$

since we have a SISO system with a positive feedback loop. The small gain theorem guarantees close-loop stability for all Δ such that

$$\gamma(\Delta)\gamma(G) < 1,$$

where γ denotes the gain. In this case we have

$$\gamma(G) = \sup_{\omega} |G(i\omega)| = 4,$$

given by the lower left amplitude plot, so the system is guaranteed stable for all Δ with $\gamma(\Delta) < K$, $K = 1/4$. We need to assume that G is a BIBO stable linear SISO system.

(b) (i) Let us use describing function analysis to predict if the system will show sustained oscillations. The describing function of a relay, $u = -\mathrm{sgn}(y)$, is

$$N(A) = \frac{4}{\pi A}.$$

The relay is an odd static non-linearity with infinite slope at the beginning so the describing function will be real and go from infinity to zero as the amplitude A increases. We get sustained oscillations if the loop-gain is one and phase $-\pi$:

$$G(i\omega)N(A) = -1,$$

i.e. if the Nyquist curve of the system intersects $-1/N(A)$. In our case $-1/N(A)$ covers the negative real axis. The Nyquist curve of the system, shown in Fig. 1, clearly intersects the negative real axis at -1.25, with small A to the right and large to the left. The describing function analysis therefore predicts stable oscillations. The frequency of the intersection is given by

$$G(i\omega) = \frac{-\omega i}{1-\omega^2+0.8\omega i} = \frac{-\omega i(1-\omega^2-0.8\omega i)}{(1-\omega^2)^2+0.64\omega^2}$$
$$= \frac{-0.8\omega^2}{\omega^4-1.36\omega^2+1} - \frac{(1-\omega^2)\omega i}{\omega^4-1.36\omega^2+1},$$

when the imaginary part is zero. This occur for $\omega_0 = 0$ and $\omega_1 = 1$. The first corresponds to the origin and the later to the interesting intersection. The predicted period is $T = \frac{2\pi}{\omega_1} = 2\pi$. The predicted amplitude is given by

$$G(i\omega_1) = -\frac{1}{N(A)} = -\frac{\pi A}{4} \Rightarrow A = -\frac{4G(i\omega_1)}{\pi} = \frac{5}{\pi} \approx 1.59.$$

Note that the describing function analysis does not provide sufficient nor necessary conditions for sustained oscillations, hence this is only a prediction that needs to be verified by for example simulations. Also note that the frequency and amplitude at the intersection at the origin is zero so it does not correspond to any sustained oscillations, moreover, if it did then they would be unstable.

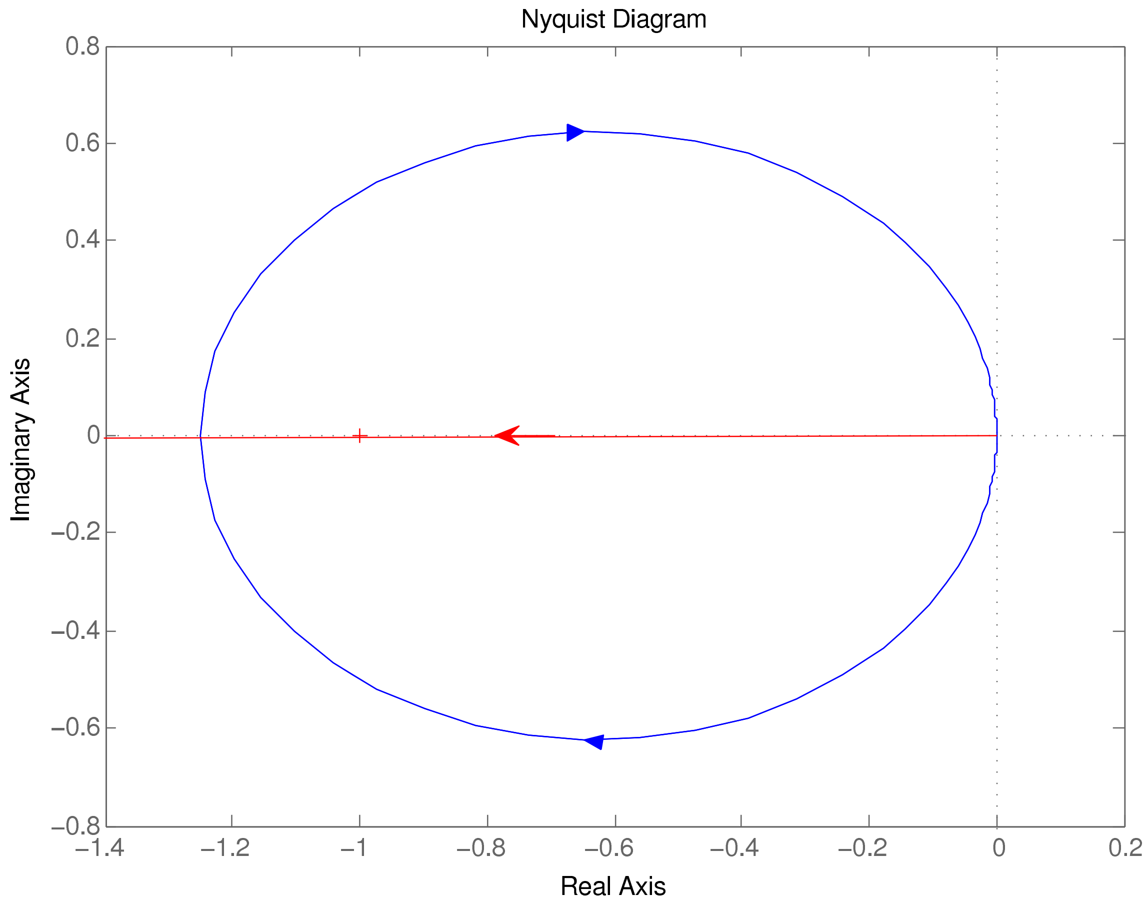

Figure 1: The Nyquist curve of $G(s) = \frac{-s}{s^2+0.8s+1}$ (blue) and the describing function of a relay (red line).

(ii) One can add an integration in the feedback loop, which gives an additional phase of -90^o, so that information about the frequency where $\arg(G(i\omega)) = -90^o$ is obtained.

5. We shall consider sliding mode control of the system

$$\dot{x}_1 = -2x_1 - \frac{x_2}{1+x_2^2} + u$$
$$\dot{x}_2 = \frac{x_1}{1+x_1^2}$$

(a) The dynamics on the sliding manifold is obtained by insertion of $x_1 = x_2$ in the second state equation:

$$\dot{x}_2 = \frac{x_2}{1+x_2^2}.$$

This system is clearly unstable, since if x_2 is positive then it will increase, while it will decrease if x_2 is negative. In other words, every trajectory that reaches the sliding manifold will diverge away from the origin along the sliding manifold. The speed of divergence will however decrease as $|x_2|$ increases.

(b) Let us introduce $\sigma(x) := x_1 + ax_2$, and not that (a) is a special case of this sliding manifold with $a = -1$. The dynamics on the sliding manifold is obtained by insertion of $x_1 = -ax_2$ in the second state equation:

$$\dot{x}_2 = \frac{-ax_2}{1+a^2x_2^2}. \tag{1}$$

Since $\text{sign}(\dot{x}_2) = -a\text{sign}(x_2)$, $a < 0$ implies that (1) is unstable and every trajectory reaching the sliding manifold will diverge from the origin along it. If $a = 0$ then every point on the sliding manifold is a stable equilibrium, i.e. every trajectory that reaches the sliding manifold will remain at the point where it reached the sliding manifold. If $a > 0$ then (1) is stable and every trajectory reaching the sliding manifold will approach the origin along it. Note that since

$$\dot{x}_1 = \frac{-ax_1}{1+x_1^2}$$

we get faster convergence (or divergence) for larger magnitudes of a.

(c) We want to make the sliding manifold S a globally attracting invariant set. This can be done with the Lyapunov function $V(x) = 0.5\sigma^2$, with $\sigma(x) := x_1 + ax_2$. This Lyapunov function is clearly continuously differentiable for $x \in \mathbb{R}^2$, positive definite and only zero on the sliding manifold, $\sigma(x) = 0$, as well as radially unbounded. Let us now design a controller such that $\dot{V} < 0 \; \forall \sigma(x) \neq 0$. The derivative is

$$\dot{V}(x) = \sigma\dot{\sigma} = \sigma(\dot{x}_1 + a\dot{x}_2) = \sigma(-2x_1 - \frac{x_2}{1+x_2^2} + u + \frac{ax_1}{1+x_1^2}),$$

so if we select the sliding mode controller

$$u = 2x_1 + \frac{x_2}{1+x_2^2} - \frac{ax_1}{1+x_1^2} - \text{sign}(x_1 + ax_2)$$

then

$$\dot{V}(x) = -\sigma\text{sign}(x_1 + ax_2) = -|\sigma| < 0 \; \forall \sigma(x) \neq 0.$$

Hence the sliding manifold is globally asymptotically stable according to Lyapunovs theorem for global asymptotic stability.
The equivalent control is computed from $\dot{\sigma}(x) = 0$, with $\sigma(x) = 0$:

$$\dot{\sigma}(x) = \dot{x}_1 + a\dot{x}_2 = -2x_1 - \frac{x_2}{1+x_2^2} + u_{eq} + \frac{ax_1}{1+x_1^2} = 0$$
$$\Rightarrow \; u_{eq} = 2x_1 + \frac{x_2}{1+x_2^2} - \frac{ax_1}{1+x_1^2}.$$