AUTOMATIC CONTROL

KTH

Nonlinear Control, EL2620 / 2E1262

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1. Consider the system

$$\dot{x}_1 = x_1(1 - x_1 - x_2)$$
$$\dot{x}_2 = x_2(x_1 - x_2) + u$$

(a) The equilibriums are given by:

$$x_1(1 - x_1 - x_2) = 0$$
$$x_2(x_1 - x_2) = 0,$$

which gives: $(0,0), (1,0), (0.5, 0.5)$. The Jacobian is

$$A(x_1, x_2) = \begin{pmatrix} 1 - 2x_1 - x_2 & -x_1 \\ x_2 & -2x_2 + x_1 \end{pmatrix}.$$

Insertion of the origin and calculation of the eigenvalues gives

$$\det(\lambda I - A(0,0)) = \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{vmatrix} = \lambda^2 - \lambda = 0$$
$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 1$$

The real part of the first eigenvalue is zero, so the equilibrium is non-hyperbolic and linear analysis is insufficient. Insertion of $(\epsilon, 0)$, with $\epsilon > 0$ small yields $\dot{x}_1 = \epsilon - \epsilon^2 \approx \epsilon$, so the origin is an unstable equilibrium.
Insertion of $(1,0)$ and calculation of the eigenvalues gives

$$\det(\lambda I - A(1,0)) = \begin{vmatrix} \lambda + 1 & 1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 1) = 0$$
$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$$

The real parts are not zero, so the equilibrium is hyperbolic and the linear analysis is valid locally. The equilibrium is a saddle point, which is unstable.
Insertion of $(0.5, 0.5)$ and calculation of the eigenvalues gives

$$\det(\lambda I - A(0.5,0.5)) = \begin{vmatrix} \lambda + 0.5 & 0.5 \\ -0.5 & \lambda + 0.5 \end{vmatrix} = \lambda^2 + \lambda + 0.5 = 0$$
$$\Rightarrow \lambda = \frac{-1}{2} \pm \frac{1}{2}\sqrt{1^2 - 2} = -\frac{1}{2} \pm \frac{i}{2}$$

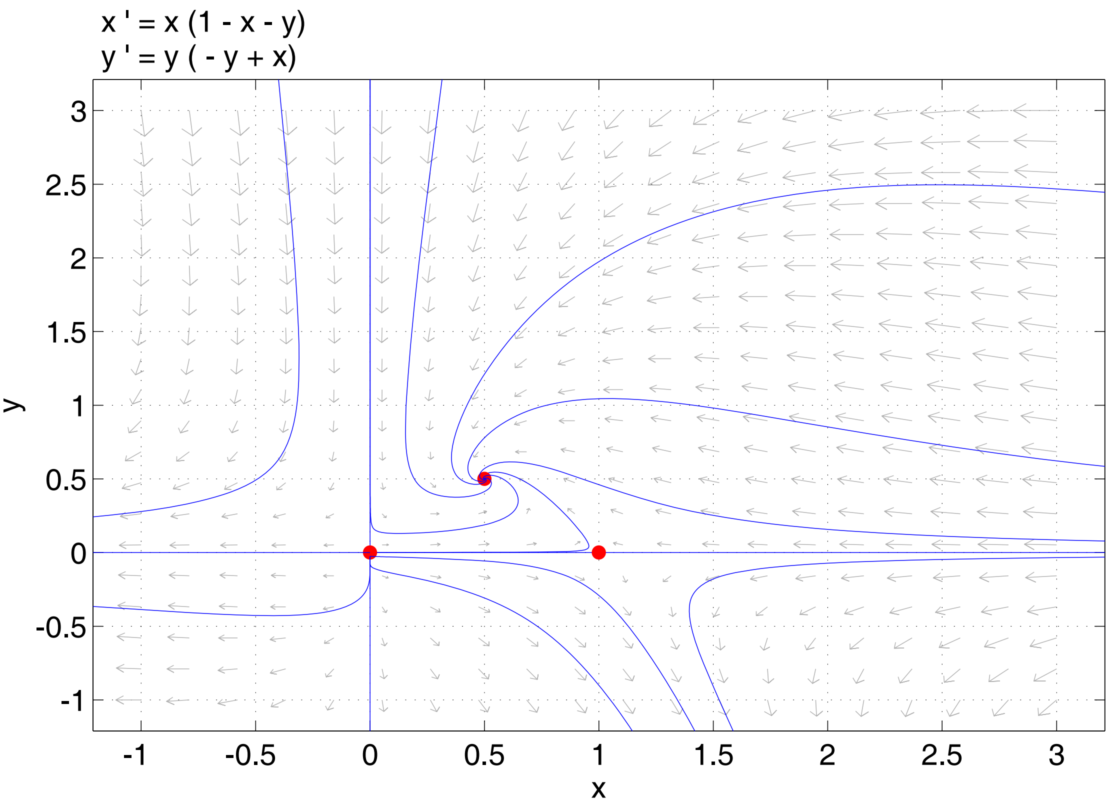

Figure 1: The phase portrait for $u = 0$.

The real parts are not zero, so the equilibrium is hyperbolic and the linear analysis is valid locally. The equilibrium is a stable focus, which is asymptotically stable.

Three equilibria exists, so none of them can be globally stable.

(b) Assuming that both species are introduced they will after some time converge towards the equilibrium at $(0.5, 0.5)$. If only the plant eater is introduced then it will converge to the equilibrium $(1, 0)$.

(c) Introduction of the general linear state feedback $u = k_1x_1 + k_2x_2$, gives the system

$$\dot{x}_1 = x_1(1 - x_1 - x_2)$$
$$\dot{x}_2 = x_2(x_1 - x_2 + k_2) + k_1x_1.$$

In order to have an equilibrium the second equation gives the requirement $x_2(x_1 - x_2 + k_2) + k_1x_1 = 0$. Insertion of $(1, 0)$ in it gives $k_1 = 0$, so we must select $k_1 = 0$, i.e. the landowner may only hunt meat eaters. We henceforth only consider hunt of meat eaters. The system has four equilibria: $(0, 0), (1, 0), (0, k_2), (\frac{1-k_2}{2}, \frac{1+k_2}{2})$. Global stability of $(1, 0)$ is thus impossible. We note that only the later two equilibria are affected by k_2. They need to be made physically impossible, i.e. negative, which gives the condition $k_2 < -1$. In addition $(1, 0)$ should be made locally asymptotically stable and

$(0,0)$ unstable. Let us use Lyapunov's indirect method. The Jacobian is

$$A(x_1, x_2) = \begin{pmatrix} 1 - 2x_1 - x_2 & -x_1 \\ x_2 & -2x_2 + x_1 + k_2 \end{pmatrix}.$$

Insertion of $(1,0)$ and calculation of the eigenvalues gives

$$\begin{aligned} \det(\lambda I - A(1,0)) = & \begin{vmatrix} \lambda + 1 & 1 \\ 0 & \lambda - 1 - k_2 \end{vmatrix} = (\lambda + 1)(\lambda - 1 - k_2) = 0 \\ \Rightarrow & \lambda_1 = 1 + k_2,\ \lambda_2 = -1 \end{aligned}$$

The real parts are nonzero if $k_2 \neq -1$, so the equilibrium is hyperbolic and the linear analysis is valid locally. If $k_2 < -1$ then the equilibrium is a stable node. Insertion of $(0,0)$ and calculation of the eigenvalues gives

$$\begin{aligned} \det(\lambda I - A(1,0)) = & \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - k_2 \end{vmatrix} = (\lambda - 1)(\lambda - k_2) = 0 \\ \Rightarrow & \lambda_1 = k_2,\ \lambda_2 = 1 \end{aligned}$$

The real parts are nonzero if $k_2 \neq 0$, so the equilibrium is hyperbolic and the linear analysis is valid locally and it is unstable. In particular, if $k_2 < -1$ then the equilibrium is a saddle point. We can hence select any $k_2 < -1$, for example $u = -2x_2$. Note that the region of attraction covers the physically possible set $\{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0\}$ except for the origin.

2. (a) (i) No, it is impossible for any trajectory to go from the interior point A of the stable limit cycle to the exterior point B, since the limit cycle constitutes the closure of an invariant set.

   (ii) No, it is impossible for any trajectory to go from the interior point A of the unstable limit cycle to the exterior point B, since the limit cycle constitutes the closure of an invariant set.

   (b) Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^2 + x_2 \\ \dot{x}_2 &= -x_1 x_2 + u \\ y &= x_2 \end{aligned}$$

   (i) A standard choice for transformation is $z_1 = x_i, z_2 = \dot{x}_i$. To avoid u being included in the definition of our new state variables, we try $z_1 = x_1$, $z_2 = -x_1^2 + x_2$. Then $x_1 = z_1$ and $x_2 = z_1^2 + z_2$. The system can then be written as

$$\begin{aligned} \dot{z}_1 &= \dot{x}_2 = z_2 \\ \dot{z}_2 &= -2x_1\dot{x}_1 + \dot{x}_2 = +2x_1^3 - 3x_1x_2 + u \; (= -z_1^3 - 3z_1z_2 + u). \end{aligned}$$

   Now the state feedback

$$\begin{aligned} u &= -2x_1^3 + 3x_1x_2 - v \\ v &= l_1 z_1 + l_2 z_2 \end{aligned}$$

will linearize the system by canceling out the nonlinearities giving

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= -l_1 z_1 - l_2 z_2.\end{aligned}$$

Note that this feedback linearization works everywhere in the state space.

(ii) We first differentiate the output until we encounter the input:

$$\begin{aligned}y &= x_2\\ \dot{y} &= \dot{x}_2 = -x_1 x_2 + u.\end{aligned}$$

The input

$$\begin{aligned}u &= +x_1 x_2 - v\\ v &= l_1 x_1 + l_2 x_2\end{aligned}$$

linearizes the input-output behaviour everywhere in the state space. The input affects the first derivative of y everywhere in the state space, so this system has a strong relative degree of 1. The relative degree is lower than the number of states so unobservable zero dynamics exist. When the output and all its derivatives are zero

$$\begin{aligned}y &= x_2 = 0 \ \Rightarrow x_2 = 0\\ \dot{y} &= v = 0 \ \Rightarrow v = 0.\end{aligned}$$

then the zero dynamics is governed by

$$\dot{x}_1 = -x_1^2 < 0 \ \forall x_1 \neq 0$$

and thus unstable.

3. (a) Lyapunov redesign of the nonlinear system

$$\begin{aligned}\dot{x}_1 &= 5x_1 x_2\\ \dot{x}_2 &= 2x_1^5 + 3u\end{aligned}$$

We need to design a state feedback such that the origin becomes globally asymptotically stable. The Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is continuously differentiable for $x \in \mathbb{R}^2$, positive definite and only zero at the origin, as well as radially unbounded. So if we can choose u such that its time derivative is negative except at the origin, then Lyapunovs theorem for global asymptotic stability is fulfilled. The time derivative is

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = 5x_1^2 x_2 + 2x_1^5 x_2 + 3x_2 u$$

so let us select

$$u = -\frac{1}{3x_2}(5x_1^2 x_2 + 2x_1^5 x_2 + x_2^2) = -\frac{5}{3}x_1^2 - \frac{2}{3}x_1^5 - \frac{1}{3}x_2$$

then we have

$$\dot{V}(x) = -x_2^2.$$

Unfortunately $\dot{V}(x) = 0$ when $x_2 = 0$, i.e. on the whole x_1-axis, so Lyapunovs theorem for global asymptotic stability is not fulfilled. Let us instead use LaSalle's theorem for global asymptotic stability, i.e. we additionally need to show that the origin is the only invariant set on the x_1-axis. On the x_1-axis the system, including the controller, reduces to

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -5x_1^2 \neq 0 \; \forall x_1 \neq 0, \end{aligned}$$

hence the origin constitutes the only invariant set on it. LaSalle's theorem now guarantees global asymptotic stability. Note that we cannot add the additional term $-\frac{x_1^2}{3x_2}$ needed to use Lyapunovs theorem to the controller, since we then would have an infinite control signal at the origin. The selected controller is finite and works in the whole state space.

(b) Consider the system

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)^2 + (x_1 - x_2)x_2 + u \; (= x_1^2 - x_1 x_2 + u) \\ \dot{x}_2 &= -x_2 + u \end{aligned}$$

(i) Back stepping cannot be applied directly since the input affects both states and this system is hence not on strict feedback form. The states need to be selected such that the effect of the input is mediated through a cascade of other states, except for one state.

(ii) We need to transform the system to strict feedback form so the backstepping procedure can be used to stabilize the states of the system one by one. Let us introduce $z_1 = x_1 - x_2$, since $\dot{z}_1 = \dot{x}_1 - \dot{x}_2$ then is independent of the input, and $z_2 = x_2$, so that the system can be written as

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 - \dot{x}_2 = x_1^2 - x_1 x_2 + x_2 = (x_1 - x_2)^2 + (x_1 - x_2 + 1)x_2 \\ &= z_1^2 + (z_1 + 1)z_2 \\ \dot{z}_2 &= -z_2 + u. \end{aligned} \tag{1}$$

The inverse state transformation is $x_1 = z_1 + z_2$ and $x_2 = z_2$. Note that the selected state transformation is clearly continuously differentiable and invertible with a continuously differentiable inverse in $x \in \mathbb{R}^2$, so it constitutes a diffeomorphism. The behaviour of the transformed system is therefore identical to the behaviour of the original one.
Let us start by only considering the first equation, regarding $z_2 = \phi_1(z_1)$ as a input to be designed:

$$\dot{z}_1 = z_1^2 + (z_1 + 1)\phi_1(z_1). \tag{2}$$

Let us now use our favourite Lyapunov function $V_1(z_1) = 0.5z_1^2$ to design $\phi_1(z_1)$ such that the system (2) becomes globally asymptotically stable.

This Lyapunov function is clearly continuously differentiable for $z_1 \in \mathbb{R}$, positive definite and only zero at the origin, as well as radially unbounded. The derivative is

$$\dot{V}_1(z_1) = z_1\dot{z}_1 = z_1^3 + (z_1^2 + z_1)\phi_1(z_1),$$

so if we for example select

$$\phi_1(z_1) = -\frac{1}{z_1^2 + z_1}(z_1^3 + z_1^2) = -z_1,$$

then $\dot{V}_1(z_1) = -z_1^2 < 0\ \forall z_1 \neq 0$ and Lyapunovs theorem for global asymptotic stability is fulfilled.

The system (1) can now be rewritten as

$$\begin{aligned}
\dot{z}_1 &= z_1^2 + (z_1+1)\phi_1(z_1) + (z_1+1)[z_2 - \phi_1(z_1)] \\
\dot{z}_2 &= -z_2 + u.
\end{aligned}$$

Let us next do a backstep by introducing $\xi_1 = z_2 - \phi_1(z_1)$ as a new state variable, and rewrite the system as

$$\begin{aligned}
\dot{z}_1 &= z_1^2 + (z_1+1)\phi_1(z_1) + (z_1+1)\xi_1 \\
\dot{\xi}_1 &= \dot{z}_2 - \dot{\phi}_1(z_1) = -z_2 + u + \dot{\phi}_1(z_1) = v,
\end{aligned}$$

where we also introduced the new controller $v = -z_2 + u - \dot{\phi}_1(z_1)$. Here we want to design v such that the whole system is globally asymptotically stable. If we select the Lyapunov function $V_2(z_1, \xi_1) = V_1(z_1) + 0.5\xi_1^2$, then the backstepping lemma ensures that a stabilizing state feedback law exists. This Lyapunov function is clearly continuously differentiable for $[z_1\ \xi_1]^T \in \mathbb{R}^2$, positive definite and only zero at the origin, as well as radially unbounded. The derivative is

$$\begin{aligned}
\dot{V}_2(z_1, \xi_1) &= \frac{dV_1}{dz_1}\dot{z}_1 + \xi_1\dot{\xi}_1 = \frac{dV_1}{dz_1}(z_1^2 + (z_1+1)\phi_1(z_1) + (z_1+1)\xi_1) + \xi_1 v \\
&= \dot{V}_1(z_1) + \frac{dV_1}{dz_1}(z_1+1)\xi_1 + \xi_1 v = -z_1^2 + z_1(z_1+1)\xi_1 + \xi_1 v,
\end{aligned}$$

so if we for example select

$$v = -\frac{1}{\xi_1}(z_1(z_1+1)\xi_1 + \xi^2) = -z_1(z_1+1) - \xi_1,$$

then $\dot{V}_2(z_1, \xi_1) = -z_1^2 - \xi^2 < 0\ \forall [z_1\ \xi_1] \neq [0\ 0]$ and Lyapunovs theorem for global asymptotic stability is fulfilled. The origin of (1) is hence globally asymptotically stable with the state feedback control law

$$\begin{aligned}
u &= z_2 + \dot{\phi}_1(z_1) + v = z_2 + \dot{\phi}_1(z_1) - z_1(z_1+1) - \xi_1 \\
&= -z_1(z_1+1) + \phi_1(z_1) + \dot{\phi}_1(z_1) \\
&== -z_1^2 - \dot{z}_1 = -2z_1^2 - (z_1+1)z_2.
\end{aligned}$$

Note that the control signal remains finite for all $z \in \mathbb{R}^2$, so the origin is indeed a globally asymptotically stable equilibrium of the closed-loop system. The origin of the original state-space is mapped by the state transformations to the origin of the new state-space, so we have actually stabilized the origin of the original system.

4. (a) We have a stable linear system $G(s)$ with a negative feedback loop containing a static nonlinearity with gain $\gamma(\phi)$.

   (i) In this case $\gamma(\phi) = k$. The small gain theorem then guarantees close-loop BIBO stability for all k such that

$$\gamma(\phi)\gamma(G) < 1.$$

   In this case we have from the Nyquist diagram

$$\gamma(G) = \sup_{\omega} |G(i\omega)| \approx 3.35,$$

   so the system is guaranteed stable for all k with $k < \frac{1}{3.35} \approx 0.3$.

   (ii) In this case the static nonlinearity is bounded by $0 < \phi(y)/y < \alpha$ and $G(s)$ has no poles in the right half plane. The circle criterion then guarantees closed-loop BIBO stability if the Nyquist curve of $G(s)$ does encircle or intersect the circle defined by $-\infty$ and $-1/\alpha$, i.e. it does not intersect the line $-1/\alpha = -0.5$. We must require $\alpha \leq 2$.

   (iii) In this case the static nonlinearity is bounded by $\beta < \phi(y)/y < 0$, which is equivalent to $0 < -\phi(y)/y < -\beta$, so we actually have a positive feedback loop. The linear system $G(s)$ has no poles in the right half plane, so the circle criterion then guarantees closed-loop BIBO stability if the Nyquist curve of $G(s)$ is enclosed by the circle defined by the points $-\infty$ and $-1/\beta$, i.e. it should not intersect the line $-1/\beta$. The right most point of the Nyquist curve is approximately 3.25, so we must require $0 > \beta \geq -1/3.25 \approx -0.3$.

(b) We are asked to use describing function analysis to predict if the system has a limit cycle, i.e. sustained oscillations. The describing function is

$$N(A) = A + 3A^2,$$

so we are dealing with an odd static non-linearity and $N(A)$ increases as the amplitude A increases. We get sustained oscillations if the loop-gain is one and phase $-\pi$:

$$G(i\omega)N(A) = -1,$$

i.e. if the Nyquist curve of the system intersects $-1/N(A)$. In our case $-1/N(A)$ covers the negative real axis. The Nyquist curve of the system, shown in Fig. 2, clearly intersects the negative real axis at -0.245, with small A to the left and large to the right (indicated by the arrow). The describing function analysis therefore predicts unstable oscillations. The frequency of the intersection is given by

$$\begin{aligned}
G(i\omega) &= \frac{1}{(i\omega+1)^4} = \frac{(-i\omega+1)^4}{(\omega^2+1)^4} = \frac{(-\omega^2-2i\omega+1)^2}{(\omega^2+1)^4} \\
&= \frac{\omega^4-6\omega^2+1}{(\omega^2+1)^4} + i\frac{4\omega(\omega^2-1)}{(\omega^2+1)^4},
\end{aligned}$$

when the imaginary part is zero. This occur for $\omega_0 = 0$ and $\omega_1 = 1$. The first corresponds to the origin and the later to the interesting intersection. The predicted period is $T = \frac{2\pi}{\omega_1} = 2\pi$. The predicted amplitude is given by

$$G(i\omega_1) = -\frac{1}{N(A)} = -\frac{1}{A+3A^2} \Rightarrow 3G(i\omega_1)A^2 + G(i\omega_1)A + 1 = 0$$

with $G(i\omega_1) = -0.25$

$$A^2 + \frac{1}{3}A - \frac{4}{3} = 0 \Rightarrow A = -\frac{1}{6} \pm \frac{7}{6}$$

as $A = 1$. Note that the describing function analysis does not provide sufficient nor necessary conditions for sustained oscillations, hence this is only a prediction that needs to be verified by for example simulations. Also note that the frequency at the intersection at the origin is zero and amplitude is $\frac{-1\pm i\sqrt{11}}{6}$, so it does not correspond to any sustained oscillations.

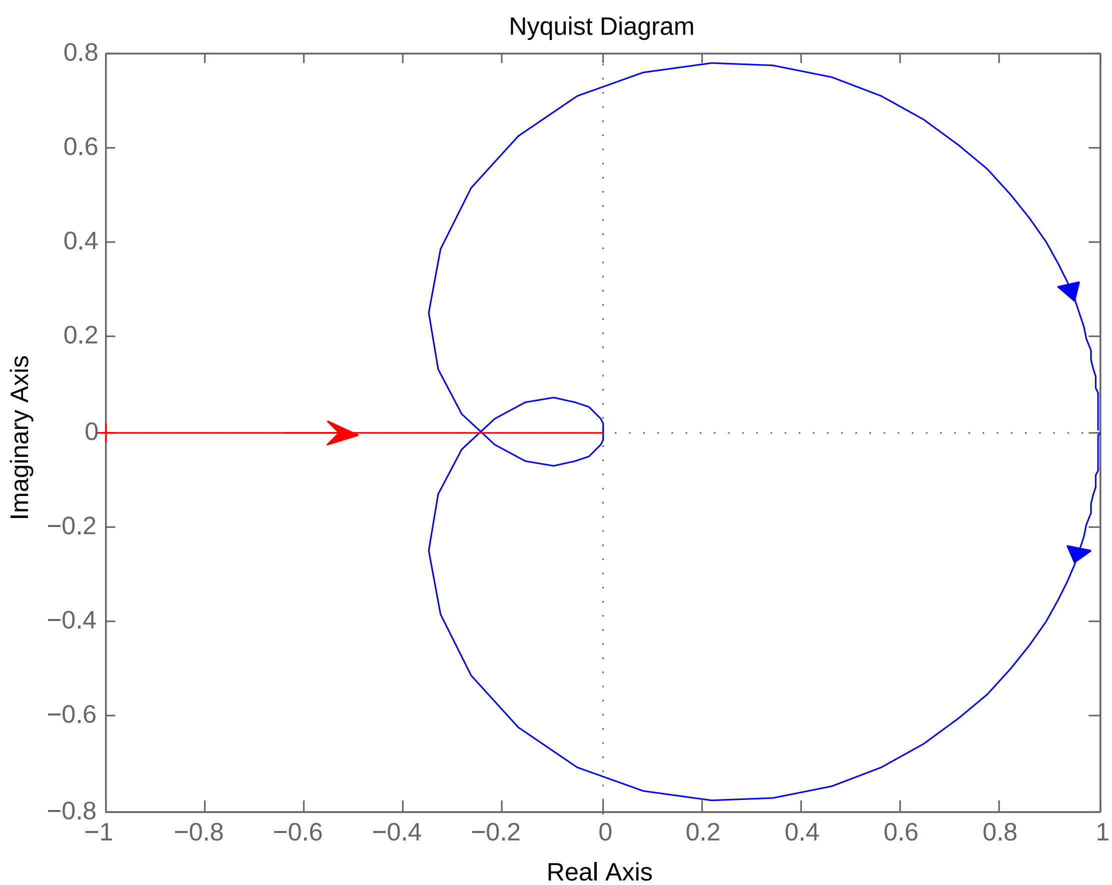

Figure 2: The Nyquist curve of $G(s) = \frac{1}{(s+1)^4}$ (blue) and the describing function $-1/N(A) = \frac{-1}{A+3A^2}$ (red line).

5. Optimal control

(a) Heating problem formulated as an optimal control problem, using given variables

$$\begin{aligned} \min_{Q(t)} \; & \int_0^{t_f} Q(t)^2 dt \\ \text{s.t.} \; & \frac{dT}{dt}(t) = -\frac{\epsilon}{c_P}(T(t) - 20^oC) + \frac{1}{c_P}Q(t) \\ & T(0) = 20^oC, \; T(t_f) = 220^oC, \; t_f = 3600s. \end{aligned}$$

It is however preferable to convert all variables and coefficients into dimensionless quantities (no units), by scaling them with some suitable coefficients. Let us select these scaling coefficients such that all stated initial value conditions become zero and all stated final value conditions one. We therefore introduce the coefficients $t_f = 3600s$, $Q_f = 1kW$, $T_i = 20^oC$, $T_f = 220^oC$ and variables

$$\begin{aligned} \tau &= \tfrac{t}{t_f} & & t &&= t_f\tau = \tau 3600s \\ u(\tau) &= \tfrac{Q(t)}{Q_f} & \Leftrightarrow \quad & Q(t) &&= Q_f u(\tau) = u(\tau)1kW \\ x(\tau) &= \tfrac{T(t)-T_i}{T_f-T_i} & & T(t) &&= (T_f - T_i)x(\tau) + T_i = x(\tau)200^oC + 20^oC. \end{aligned}$$

The heat equation can now be rewritten as

$$\begin{aligned} \frac{dx}{d\tau}(\tau) &= \frac{1}{T_f - T_i}\frac{dT}{dt}\frac{dt}{d\tau} = \underbrace{-\frac{\epsilon t_f}{c_P}}_{=a}\underbrace{\frac{T(t) - T_i}{T_f - T_i}}_{=x(\tau)} + \underbrace{\frac{t_f Q_f}{c_P(T_f - T_i)}}_{=b} u(\tau) \\ x(0) &= 0, \; x(1) = 1, \end{aligned}$$

where we introduced the new dimensionless coefficients $a = -\frac{\epsilon t_f}{c_P} = -36$ and $b = \frac{t_f Q_f}{c_P(T_f - T_i)} = 1800$. And the objective can be written as

$$\Xi = \min_{Q(t)} \int_0^{t_f} Q(t)^2 dt = \min_{u(\tau)} \int_0^1 u(\tau)^2 Q_f^2 t_f d\tau = Q_f^2 t_f \underbrace{\min_{u(\tau)} \int_0^1 u(\tau)^2 d\tau}_{=\Theta}.$$

We may as well minimise Θ since the original objective merely is a constant times it, i.e. $\Xi = Q_f^2 t_f \Theta = \Theta 3600k^2W^2s = \Theta MW^2h$. The optimal control problem in dimensionless quantities is hence

$$\begin{aligned} \min_{u(t)} \; & \int_0^1 u(\tau)^2 d\tau \\ \text{s.t.} \; & \frac{dx}{d\tau}(\tau) = ax(\tau) + bu(\tau) \\ & x(0) = 0, \; x(1) = 1. \end{aligned} \tag{3}$$

(b) With no heat loss it is trivial to see that we need exactly $Q_{tot} = 200^oC \cdot 1kJ/^oC = 200kJ$ to bring the oven from 20^oC to 220^oC. When or how we heat the oven would not matter if it was not for $Q(t)$ being squared in the objectivity function. The square implies that it is optimal to use constant heating with $Q(t) = Q_{tot}/t_f = \frac{200kJ}{3600s} = \frac{1}{18}kW$.

(c) We use the general Pontryagin's maximum principle.

- First we identify the necessary functions from (3):

$$\begin{aligned} L(u(\tau)) &= u(\tau)^2,\ \phi = 0 \\ f(x(\tau), u(\tau)) &= ax(\tau) + bu(\tau) \\ \Psi(x(\tau_f)) &= x(1) - 1. \end{aligned}$$

- Second we form the Hamiltonian

$$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T f(x, u) = n_0 u(\tau)^2 + a\lambda(\tau)x(\tau) + b\lambda(\tau)u(\tau).$$

- Third we minimize the Hamiltonian in order to find the optimal $u^*(\tau)$

$$H(x^*, u^*, \lambda, n_0) = \min_{u(\tau)} H(x^*, u, \lambda, n_0).$$

We have no constraint on the input so we will find the optimum where the derivative is zero

$$\frac{\partial H}{\partial u}(x^*, u, \lambda, n_0) = 2n_0 u(\tau) + b\lambda(\tau) \equiv 0 \ \Rightarrow u^*(\tau) = -\frac{b\lambda(\tau)}{2n_0}.$$

In order to ensure that we obtain a minimum we must require that the second derivative is positive

$$\frac{\partial^2 H}{\partial u^2}(x^*, u, \lambda, n_0) = 2n_0 > 0 \ \Rightarrow n_0 > 0.$$

Note that the requirement $(n_0, \mu^T) \neq 0$ is hence already fulfilled and μ may take any value.

- Fourth we solve the adjoint and system equations to obtain $\lambda(\tau)$ and $x(\tau)$. The adjoint equations are

$$\begin{aligned} \dot{\lambda}(\tau) &= -\frac{\partial H}{\partial x}(x^*, u^*, \lambda, n_0) = -a\lambda(\tau) \\ \lambda(\tau_f) &= \mu. \end{aligned}$$

with solution

$$\lambda(\tau) = Ae^{-a\tau},\ A = \mu e^{a\tau_f} = \mu e^a \ \Rightarrow \lambda(\tau) = \mu e^{a(1-\tau)}.$$

Note that the final value condition merely enables us to express the constant A using the other unknown constant μ.
The system equations are

$$\begin{aligned} \frac{dx}{d\tau}(\tau) &= ax(\tau) + bu^*(\tau) = ax(\tau) - \frac{b^2 e^a}{2}\frac{\mu}{n_0}e^{-a\tau} \\ x(0) &= 0,\ x(1) = 1, \end{aligned}$$

with solution

$$x(\tau) = \frac{b^2 e^a}{4a}\frac{\mu}{n_0}e^{-a\tau} + Be^{a\tau}.$$

Note that both μ and n_0 are unknown at this point and that we only will be able to determine the ratio of them from the initial and final value conditions

$$\frac{b^2e^a}{4a}\frac{\mu}{n_0} + B = 0 \quad \Rightarrow B = -\frac{b^2e^a}{4a}\frac{\mu}{n_0}$$
$$\frac{b^2e^a}{4a}\frac{\mu}{n_0}e^{-a} + Be^a = 1 \quad \Rightarrow \frac{b^2e^a}{4a}\frac{\mu}{n_0}\left(e^{-a} - e^a\right) = 1.$$

Let us now introduce

$$\xi = \frac{1}{e^{-a} - e^a} = \frac{1}{e^{36} - e^{-36}} \approx 2.32 \cdot 10^{-16},$$

since we then have $\frac{\mu}{n_0} = \frac{4a}{b^2e^a}\xi \approx -4.44 \cdot 10^{-5}$ and $B = -\xi$. The optimal scaled control signal is thus

$$u^*(\tau) = -2\frac{a}{b}\xi e^{-a\tau} = -2\frac{a}{b}\frac{e^{-a\tau}}{e^{-a} - e^a} = 0.04\frac{e^{36\tau}}{e^{36} - e^{-36}}$$

and the optimal scaled temperature is

$$x^*(\tau) = \xi(e^{-a\tau} - e^{a\tau}) = \frac{e^{-a\tau} - e^{a\tau}}{e^{-a} - e^a} = \frac{e^{36\tau} - e^{-36\tau}}{e^{36} - e^{-36}}.$$

(d) The heat loss when the temperature of the oven has reached 220^oC is $\frac{dx}{dt} = -\frac{\epsilon}{c_P}200^oC = -0.01kW/^oC \cdot 200^oC = -2kW$, so with $Q_{\max} = 2kW$ we can barely maintain the required temperature. This temperature can therefore not be reached in finite time and we conclude that the optimal control problem is infeasible.