Answers/solutions to the exam in Nonlinear Control
EL2620, 2010-12-17

1. (a) Equilibria for $\dot{x} = 0$, i.e., for $x_2 = 0$ and $\sin x_1 = 0$, or $x_1 = k\pi$ where k is an integer. Linearization about equilibria x^* yields

$$A = \begin{pmatrix} 0 & 1 \\ -\cos(x_1^*) & 4x_2^{3*} - 3x_2^2 \end{pmatrix}$$

and we get the characteristic equation $\lambda^2 = 1$ when k is odd and $\lambda^2 = -1$ when k is even. Thus, we have eigenvalues $-1, 1$ for k odd, corresponding to a saddle point, and eigenvalues $\pm i$ for k even, corresponding to a center for the linearized system. The equilibria for k odd are hyperbolic with positive eigenvalues, hence locally unstable, while we the equilibria for k even are non-hyperbolic and hence we can not deduce the local stability from the linearization.

(b) Let us consider some small region Ω around the origin, e.g. $\Omega = \{(x_1, x_2) | x_1^2 + x_2^2 < 1\}$. Within Ω we have that $V \geq 0$ and $V = 0$ only at the origin. Furthermore

$$\dot{V} = \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_2 \sin x_1 - x_2 \sin x_1 + x_2^4(x_2 - 1) = x_2^4(x_2 - 1)$$

which is negative semidefinite in Ω and $\dot{V} = 0$ only when $x_2 = 0$. Thus, Ω is invariant. From the differential equation for x_2 when $x_2 = 0$, $\dot{x}_2 = -\sin x_1$ and hence the only invariant set within $x_2 = 0$ in Ω is the origin. Thus, according to LaSalle Invariant Set Theorem the origin is locally asymptotically stable.

(c) The control law $u = \sin x_1 - (x_2 - 1)x_2^3 - x_1 - 2x_2$ will make the system linear and with eigenvalues in -1. Hence, globally asymptotially stable.

2. (a) (i) with u_1 we can simply cancel the nonlinearity with $u_1 = -r_0/(K_0 + x_2^n) + v$. Note that the control becomes unbounded if $x_2^n = -K_0$, and hence we can not make the system linear in this point with the chosen control. (ii) With u_2 we need to make a state transformation first, and choose $z_1 = x_1$ and $z_2 = \dot{x}_1$, which yields

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{-n x_2^{n-1} r_0}{(K_0 + x_2^n)^2}(K_2 x_1 - K_3 x_2 + u_2) - K_1 z_1 \end{aligned}$$

Thus, by choosing $u_2 = -K_2 x_1 + K_3 x_2$ we obtain a linear system in the states z. Also, in this case is there a singularity for $x_2^n = -K_0$.

(b) (i) We have $\dot{y} = \dot{x}_2 = K_2 x_1 - K_3 x_2$, and since u_1 does not appear explicitly we have a relative degree two. Differentiating again yields $\ddot{y} = K_2 \dot{x}_1 - K_3 \dot{x}_2$ or

$$\ddot{y} = K_2 \frac{r_0}{K_0 + x_2^n} - K_2 K_1 x_1 + K_2 u_1 - K_3 K_2 x_1 + K_3^2 x_2$$

and hence the control

$$u = -\frac{r_0}{K0 + x_2^n} + (K_1 + K_3) x_1 - \frac{K3^2}{K_2} x_2 + \frac{v}{K_2}$$

yields the linear system

$$\ddot{y} = v$$

Since the relative degree is equal to the number of states there are no zero dynamics.

(ii) We have $\dot{y} = K_2 x_1 - K_3 x_2 + u_2$ and hence the relative degree is one in this case. The control $u_2 = -K_2 x_1 + K_3 x_2 + v$ yields the linear system

$$\dot{y} = v$$

In this case the zero dynamics is of order one and is trivially given by the dynamics of x_1 since x_2 is the output. From the differential equation for x_1 we see that, with x_2 constant, x_1 will diverge exponentially if $K_1 < 0$ while it will converge exponentially if $K_1 > 0$. Thus, there will be problems with unstable zero dynamics if $K_1 < 0$.

3. (a) Since $G(s)$ and k are linear functions, $\gamma(\cdot)$ must contain all nonlinearities in the system, i.e., $\gamma(\cdot) = f(\cdot)$. The input to γ is x_3 and hence the output of $G(s)$ is $y = x_3$. The input to $G(s)$ is $u = kf(x_3)$. Hence we derive

$$G(s) = \frac{1}{s(s+1)^2} \;; \quad \gamma(y) = \frac{y}{1+y^2}$$

(b) Since $G(s)$ contains a pole in $s = 0$ it is not BIBO stable, and hence the Small Gain Theorem can not be applied to deduce any information on the stability of the loop as such (of course, one may rewrite the system to apply the Small Gain Theorem, as is done in the derivation of the Circle Criterion, but that is not considered here).

(c) The nonlinearity γ is sector bounded with an upper bound $k_2 = 1$ (slope at origin) and $k_1 = 0$ (slope at infinity). Since $G(s)$ is marginally stable we can apply the Circle criterion. Note that the Circle Criterion is based on a negative feedback loop, i.e., with a -1 in the loop which is not present in the block-diagram here. The results hence apply to $-1 \cdot k$. The Circle Criterion states that $G(i\omega)$ should not have a real part less than $-1/(k_2)$. We have

$$G(i\omega) = \frac{-i/omega}{-\omega^2 + 2i\omega + 1} = \frac{-2 + \frac{i}{\omega}(\omega^2 - 1)}{4\omega^2 + (\omega^2 - 1)^2}$$

from which it is easily seen that the minimum real part of G is obtained for $\omega = 0$ for which $\Re G = -2$. Hence, the Circle Criterion gives that the system is guaranteed stable if $k > -0.5$.

(d) Since the nonlinearity is static and odd, the describing function $N(A)$ is real valued. The maximum value of $N(A) = 1$ for small A and then it decreases to $N(A) = 0$ for large A. Thus, since $G(s)$ is marginally stable, the method with describing functions will predict a sustained oscillation if $-kG(i\omega)$ crosses the real axis between $[-\infty, -1]$. From $G(i\omega)$ in (c) above, we find that G is real for $\omega = 1$, for which $-kG = k0.5$, thus we find that the Nyquist curve for G crosses $-1/N(A)$, predicting sustained oscillations, when $k < -2$

4. (a) The system can be written on the strict feedback form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x1, x_2)u \end{aligned}$$

with $f_1 = x_1^2, g_1 = x_1, f_2 = -2x_2^2$. Applying backstepping we start with $V_1(x_1) = x_1^2 \Rightarrow \dot{V}_1 = x_1\dot{x}_1 = x_1^3 + x_1^2\phi(x_1)$. By choosing $\phi(x_1) = -x_1 - x_1^2$ we get $\dot{V}_1 = -x_1^4$. Thus, with $x_2 = \phi(x_1)$ the function V_1 is a radially unbounded Lyapunov function for the first equation. The Backstepping Lemma now yields

$$u_1 = f_2 + g_2 u = \frac{d\phi}{dx_1}(f1 + g_1 x_2) - \frac{dV}{d_x 1} g_1 - (x_1 - \phi(x_1))$$

which yields the control

$$u = -2x_1^3 - 3x_1^2 - x_1x_2 - 2x_1^2x_2 - x_2 - x_1 + 2x_2^2$$

and the corresponding Lyapunov function showing global asymptotic stability of the origin is

$$V = V_1 + (x_2 - \phi(x_1))^2/2 = \frac{1}{2}(2x_1^2 + x_2 + x_1)$$

(b) (i) With $x_2 = -ax_1$ we get $\dot{x}_1 = x_1^2 - ax_1^2 = (1-a)x_1^2$, which will make x_1 (and hence x_2) diverge to infinity if $(1-a) > 0$ and $x_1(0) > 0$ or if $(1-a) < 0$ and $x_1(0) < 0$.

(ii) Considering $\dot{x}_1 = x_1(x_1 + x_2)$, we see that we can choose $x_2 = -x_1 - x_1^2$ to make $\dot{x}_1 = -x_1^3$ which will converge to $x_1 = 0$ for any initial condition. Furthemore, with $x_2 = -x_1 - x_1^2$ we get that x_2 will converge to the origin when x_1 does so. Thus, the sliding manifold

$$S = \{(x_1, x_2)|\sigma(x) = x_2 + x_1 + x_1^2 = 0\}$$

is a suitable manifold to force the system onto.
(iii) We choose the Lyapunov candidate $V = \frac{1}{2}\sigma^2 \geq 0$ which yields

$$\dot{V} = \sigma\dot{\sigma} = \sigma(\dot{x}_1(1+2x_1)+\dot{x}_2) = \sigma((1+2x_1)(x_1^2+x_1x_2)-2x_2^2+u)$$

To make $\dot{V} < 0$ except at $\sigma = 0$ we choose the control

$$u = -(1 + 2x_1)(x_1^2 + x_1x_2) + 2x_2^2 - Ksign(\sigma)$$

where $K > 0$.

5. (a) The necessary conditions are given by Pontryagin's maximum principle (lec 12, slide 8). This is an infinite time problem where the time is considered fixed. We introduce the Hamiltonian

$$H(x, w, u, \lambda) = \frac{1}{2}(x^TQx+u^TRu-\gamma^2w^Tw)+\lambda^T(Ax+B_1w+B_2u). \tag{1}$$

The first condition is

$$\min_{w,u} H(x^*, w, u, \lambda) = H(x^*, w^*, u^*, \lambda), \tag{2}$$

where the costate λ is given by the adjoint equation, which is the second condition,

$$\dot{\lambda} = -\frac{\partial H(x^*, w^*, u^*, \lambda)^T}{\partial x}, \quad \lambda(\infty) = 0. \tag{3}$$

The third condition is given by the system equation

$$\dot{x} = Ax + B_1w + B_2u.$$

We do not have any constraint on the disturbance or control input, so the optimum is given by the first order conditions

$$\frac{\partial H}{\partial u} = 0 \Leftrightarrow u = -R^{-1}B_2^T\lambda$$
$$\frac{\partial H}{\partial w} = 0 \Leftrightarrow w = \frac{1}{\gamma^2}B_1^T\lambda.$$

Insertion of these in the system equation, gives the closed-loop system

$$\dot{x} = Ax + B_1 \frac{1}{\gamma^2} B_1 B_1^T \lambda - B_2 R^{-1} B_2^T \lambda,$$

while the adjoint equation is

$$\dot{\lambda} = -Qs - A^T \lambda, \quad \lambda(\infty) = 0. \tag{4}$$

These two together gives

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & \frac{1}{\gamma^2} B_1 B_1^T - B_2 R^{-1} B_2^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{5}$$

from which we can solve x and λ.

(b) Replace λ by Px, and $\dot{\lambda}$ by $P\dot{x}$ in (4). Multiply the first equation by P and you get the requirement

$$PAx + P(\frac{1}{\gamma^2} B_1 B_1^T - B_2 R^{-1} B_2^T) Px = -Qx - A^T Px, \tag{6}$$

which only is fulfilled for all x if

$$A^T P + PA + Q + P(\frac{1}{\gamma^2} B_1 B_1^T - B_2 R^{-1} B_2^T) P = 0.$$