AUTOMATIC CONTROL

KTH

**Nonlinear Control, EL2620 / 2E1262**

Answers January 10 , 2012

1. (a) Unique equilibrium $x_1 = 0, x_2 = 0$, with Jacobian $A = [0\ 1; -1\ 1]$ and eigenvalues $\lambda(A) = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Thus, unstable focus. The local phase portrait around the origin is a outward directed spiral.

   (b) We have

   $$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 + 2x_2(-x_1 + x_2(1 - x_1^2 - 2x_2^2) = 2x_2^2(1 - x_1^2 - 2x_2^2)$$

   At the level curve $x_1^2 + x_2^2 = 0.5$ we get $\dot{V} = 2x_2^2(0.5 - x_2^2)$ and since $x_2^2 \leq 0.5$ we get that $\dot{V} \geq 0$ and hence all trajectories point outwards or parallell to the level curve. At the level curve $x_1^2 + x_2^2 = 1$ we get $\dot{V} = -4x_2^4 \leq 0$ and hence all tjactories point inwards or parallell. Thus, we have shown that the region is invariant.

   (c) Since this is a planar system and the region in (b) is invariant and does not contain any equilibrium points, according to (a), there must be a stable limit cycle within the region

2. (a) (i) Introduce $x_1 = y, x_2 = \dot{y}$. The Jacobian becomes $A = [0\ 1; -c\ \ -a-b]$. The characteristic equation is $\lambda^2 + (a+b)\lambda + c = 0$. The eigenvalues have strictly negative real part if $a > 0$ or $b > 0$ and hence the equlibrium is stable by Lyapunovs indirect method. For $a = b = 0$ we have purely imaginary eigenvalues and we therefore employ Lyapunovs direct method and try the proposed Lyapunov candidate. For $c > 0$ we have that $V geq 0$ and zero only at the equilibrium point, with $\dot{V} = c\sin x_1 x_2 + x_2(-(a + b\cos x_1)x_2 - c\sin x_1) = -x_2^2(a + b\cos x_1) \leq 0$. Hence, the equilibrium is stable is $a \geq b \geq 0$.(ii) With $a > b \geq 0$ we can show asymptotic stability using Lyapunovs indirect method. Alternatively it can be shown with the Lyapunov function above and LaSalle.

   (b) To avoid differentiation of $u$ we try the transformation $z_1 = x_2, z_2 = \dot{x}_2$. This yields $\dot{z}_1 = z_2$ and $\dot{z}_2 = \dot{x}_1 - 6x_2^2\dot{x}_2 = -x_1^2 + x_2 + u - 6x_2^2x_1 + 12x_2^5$. Now choose $u = c(x, v)$ such that $\dot{z}_2 = v$.

3. (a) The proposed Lyapunov candidate is strictly positive for $K > 0$ and all $q, \dot{q}$ except at the equilibrium $q = r, \dot{q} = 0$ where $V = 0$. The time derivative of $V$ is

   $$\dot{V} = \dot{q}M\ddot{q} + \frac{1}{2}\dot{q}^2\dot{M} + K(q - r)\dot{q}$$

   Inserting the expression for $M\ddot{q}$ and the control law, and using $\dot{M} = 2C$, we get

   $$\dot{V} = K_d\dot{q}^2$$

which is strictly negative for $K_d < 0$ except for $\dot{q} = 0$. To show asymptotic stability we consider the case when $\dot{V} = 0$, corresponding to $\dot{q} = \ddot{q} = 0$. From the equation of motion with the control law inserted we get $K(r - q) = 0$ and hence the only invariant solution for which $\dot{q} = 0$ is the equilibrium point $q = r$, and hence have asymptotic tracking according to LaSalle's theorem.

(b) With a gravitational force $g \neq 0$ we get

$$\dot{V} = K_d\dot{q}^2 - \dot{q}g(q)$$

and we can no longer guarantee $\dot{V} \leq 0$. By adding the gravitational term to the control such that

$$\tau_u = K(r - q) + K_d\dot{q} + g(q)$$

we can guarantee asymptotic tracking (but in a somewhat unrobust fashion since we require exact cancellation of the gravitational term).

(c) To make the dynamics linear we specify the closed-loop

$$\ddot{q} = \tau_v$$

which implies the control law

$$\tau_u = M(q)\tau_v + C(q, \dot{q})\dot{q} + g(q)$$

4. (a)

$$G(s) = \frac{1}{s(s^2 + s + 1)} \quad ; \quad \phi = \frac{1}{3}z^3$$

(b) The static nonlinearity is odd and hence the describing function is real. The first Fourier coefficient

$$b_1 = \frac{A^3}{3\pi}\int_0^{2\pi} \sin(\theta)^4 d\theta = \frac{A^3}{4}$$

and the describing function is

$$N(A) = \frac{A^2}{4}$$

(c) The describing function is real and hence $-1/N(A)$ resides on the negative real axis, with $-1/N(A) \to -\infty$ as $A \to 0$ and $-1/N(A) \to 0$ as $A \to \infty$. Thus, we seek frequencies for which $G(i\omega)$ is real and negative. The imaginary part of $G$ is

$$Im\ G = \frac{-(1 - \omega^2)}{\omega((1 - \omega)^2 + \omega^2}$$

and hence $\omega = 1$ gives crossing with negative real axis. We get $G(i1) = -1$, and $-1/N(A) = 1$ yields $A = 2$. Thus, we predict an oscillation with amplitude 2 and period $T = 2\pi/\omega = 6.28$. Since a descreasing $A$ will move the $-1/N(A)$ point outside the Nyquist curve, the oscillation should be unstable.

5. (a) Input-output linearization with $y = x_1$,

$$\dot{y} = \dot{x}_1 = -x_1^2 + x_1 x_2 + u$$

and hence the controller $u = x_1^2 = x_1 x_2 + v$ yields

$$Y = \frac{1}{s} V$$

Now the P=controller $V = K(R - Y)$ yields

$$Y = \frac{K}{s + K} R$$

and hence $K = 1/\tau$ yields the desired closed-loop. Since the relative order is 1 and the state dimension is 2 we have zero dynamics, corresponding to the dynamics when forcing $y = 0$. This corresponds to

$$\dot{x}_2 = x_2(x_2^2 - 1)$$

and we see that we get divergence, or instability, when $|x_2| > 1$. Thus, we may experience instability in the state $x_2$ which furthermore is unobservable in the output.

(b) On the sliding manifold $\sigma = x_1 + ax_2 = 0$ and $\dot{x}_1 = -a\dot{x}_2$. Inserting this in the differential equation for $x_2$ yields

$$\dot{x}_2 = x_2^3 + (a - 1)x_2$$

and the linear part is stable if $a < 1$. Thus, $x_2$ (and hence $x_1$) will not converge to the origin if $|x_2|$ is sufficiently large. To avoid this problem we cancel the $x_2^3$ term by employing

$$\sigma = x_1 + ax_2 + bx_2^3$$

and choose $b \geq 1$.

(c) The Lyapunov candidate $V = \frac{1}{2}\sigma^2$ yields

$$\dot{V} = \sigma(\dot{x}_1 + a\dot{x}_2 + b3x_2^2\dot{x}_2) = \sigma(-x_1^2 + x_1 x_2 + u + (a + 3bx_2^2)(x_2^3 - x_2 - x_1))$$

The control law

$$u = x_1^2 - x_1 x_2 - (a + 3bx_2^2)(x_2^3 - x_2 - x_1) - sign(\sigma)$$

ensures $\dot{V} < 0$ when $\sigma \neq 0$ and hence will make the sliding manifold globally attracting.