EL2620 Nonlinear Control

Lecture 12

- Optimal control

Today's Goal

You should be able to

- Design controllers based on optimal control theory

Optimal Control Problems

Idea: formulate the control design problem as an optimization problem

$$\min_{u(t)} J(x,u,t), \quad \dot{x} = f(t,x,u)$$

+ provides a systematic design framework
+ applicable to nonlinear problems
+ can deal with constraints

- difficult to formulate control objectives as a single objective function
- determining the optimal controller can be hard

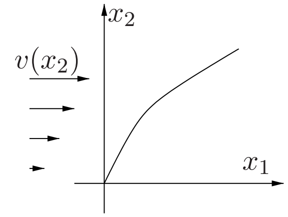
Example—Boat in Stream

Sail as far as possible in x_1 direction

Speed of water $v(x_2)$ with $dv/dx_2 = 1$

Rudder angle control:

$u(t) \in U = \{(u_1, u_2) : u_1^2 + u_2^2 = 1\}$



$$\max_{u:[0,t_f]\to U} x_1(t_f)$$

$$\dot{x}_1(t) = v(x_2) + u_1(t)$$

$$\dot{x}_2(t) = u_2(t)$$

$$x_1(0) = x_2(0) = 0$$

Example—Resource Allocation

Maximization of stored profit

$x(t) \in [0, \infty)$	production rate
$u(t) \in [0, 1]$	portion of x reinvested
$1 - u(t)$	portion of x stored
$\gamma u(t) x(t)$	change of production rate ($\gamma > 0$)
$[1 - u(t)]x(t)$	amount of stored profit

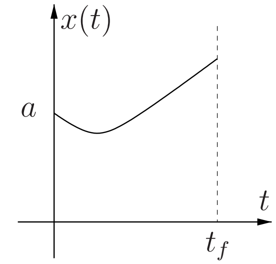
$$\max_{u:[0,t_f]\to[0,1]} \int_0^{t_f} [1-u(t)]x(t)dt$$
$$\dot{x}(t) = \gamma u(t) x(t)$$
$$x(0) = x_0 > 0$$

Example—Minimal Curve Length

Find the curve with minimal length between a given point and a line

Curve: $(t, x(t))$ with $x(0) = a$

Line: Vertical through $(t_f, 0)$



$$\min_{u:[0,t_f]\to\mathbb{R}} \int_0^{t_f} \sqrt{1 + u^2(t)}dt$$
$$\dot{x}(t) = u(t)$$
$$x(0) = a$$

Optimal Control Problem

Standard form:

$$\min_{u:[0,t_f]\to U} \int_0^{t_f} L(x(t), u(t))\, dt + \phi(x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

Remarks:

- $U \subset \mathbb{R}^m$ set of admissible control
- Infinite dimensional optimization problem:
  Optimization over *functions* $u : [0, t_f] \to U$
- Constraints on x from the dynamics
- Final time t_f fixed (free later)

Pontryagin's Maximum Principle

Theorem: Introduce the Hamiltonian function

$$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u)$$

Suppose the optimal control problem above has the solution $u^* : [0, t_f] \to U$ and $x^* : [0, t_f] \to \mathbb{R}^n$. Then,

$$\min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad \forall t \in [0, t_f]$$

where $\lambda(t)$ solves the adjoint equation

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}(x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \frac{\partial \phi^T}{\partial x}(x^*(t_f))$$

Moreover, the optimal control is given by

$$u^*(t) = \arg\min_{u \in U} H(x^*(t), u, \lambda(t))$$

Remarks

- See textbook, e.g., Glad and Ljung, for proof. The outline is simply to note that every change of $u(t)$ from the optimal $u^*(t)$ must increase the criterium. Then perform a clever Taylor expansion.
- Pontryagin's Maximum Principle provides **necessary** condition: there may exist many or none solutions
  (cf., $\min_{u:[0,1]\to\mathbb{R}} x(1)$, $\dot{x} = u$, $x(0) = 0$)
- The Maximum Principle provides all possible candidates.
- Solution involves two ODE's with boundary conditions $x(0) = x_0$ and $\lambda(t_f) = \partial\phi^T/\partial x(x^*(t_f))$. Often hard to solve explicitly.
- "maximum" is due to Pontryagin's original formulation

Example—Boat in Stream (cont'd)

Hamiltonian satisfies

$$H = \lambda^T f = \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}\begin{pmatrix}v(x_2)+u_1 \\ u_2\end{pmatrix}$$

$$\frac{\partial H}{\partial x} = \begin{pmatrix}0 & \lambda_1\end{pmatrix}, \qquad \phi(x) = -x_1$$

Adjoint equations

$$\dot{\lambda}_1(t) = 0, \qquad \lambda_1(t_f) = -1$$
$$\dot{\lambda}_2(t) = -\lambda_1(t), \qquad \lambda_2(t_f) = 0$$

have solution

$$\lambda_1(t) = -1, \quad \lambda_2(t) = t - t_f$$

Optimal control

$$u^*(t) = \arg\min_{u_1^2+u_2^2=1} \lambda_1(t)(v(x_2^*(t)) + u_1) + \lambda_2(t)u_2$$
$$= \arg\min_{u_1^2+u_2^2=1} \lambda_1(t)u_1 + \lambda_2(t)u_2$$

Hence,

$$u_1(t) = -\frac{\lambda_1(t)}{\sqrt{\lambda_1^2(t)+\lambda_2^2(t)}}, \quad u_2(t) = -\frac{\lambda_2(t)}{\sqrt{\lambda_1^2(t)+\lambda_2^2(t)}}$$

or

$$u_1(t) = \frac{1}{\sqrt{1+(t-t_f)^2}}, \quad u_2(t) = \frac{t_f - t}{\sqrt{1+(t-t_f)^2}}$$

Example—Resource Allocation (cont'd)

$$\min_{u:[0,t_f]\to[0,1]} \int_0^{t_f} [u(t)-1]x(t)dt$$
$$\dot{x}(t) = \gamma u(t)x(t), \qquad x(0) = x_0$$

Hamiltonian satisfies

$$H = L + \lambda^T f = (u-1)x + \lambda\gamma u x$$

Adjoint equation

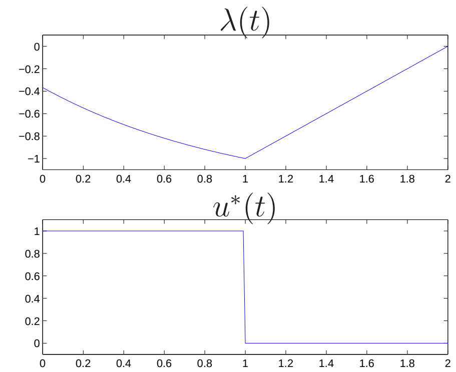
$$\dot{\lambda}(t) = 1 - u^*(t) - \lambda(t)\gamma u^*(t), \qquad \lambda(t_f) = 0$$

Optimal control

$$u^*(t) = \arg\min_{u\in[0,1]} (u-1)x^*(t) + \lambda(t)\gamma u x^*(t)$$
$$= \arg\min_{u\in[0,1]} u(1+\lambda(t)\gamma), \qquad (x^*(t) > 0)$$
$$= \begin{cases} 0, & \lambda(t) \geq -1/\gamma \\ 1, & \lambda(t) < -1/\gamma \end{cases}$$

For $t \approx t_f$, we have $u^*(t) = 0$ (why?) and thus $\dot{\lambda}(t) = 1$.

For $t < t_f - 1/\gamma$, we have $u^*(t) = 1$ and thus $\dot{\lambda}(t) = -\gamma\lambda(t)$.



- $u^*(t) = \begin{cases} 1, & t \in [0, t_f - 1/\gamma] \\ 0, & t \in (t_f - 1/\gamma, t_f] \end{cases}$
- It's optimal to reinvest in the beginning

5 minute exercise: Find the curve with minimal length by solving

$$\min_{u:[0,t_f]\to\mathbb{R}} \int_0^{t_f} \sqrt{1+u^2(t)}dt$$
$$\dot{x}(t) = u(t), \qquad x(0) = a$$

5 minute exercise II: Solve the optimal control problem

$$\min \int_0^1 u^4 dt + x(1)$$
$$\dot{x} = -x + u$$
$$x(0) = 0$$

History—Calculus of Variations

- Brachistochrone (shortest time) problem (1696): Find the (frictionless) curve that takes a particle from A to B in shortest time

  $$dt = \frac{ds}{v} = \frac{\sqrt{dx^2 + dy^2}}{v} = \frac{\sqrt{1 + y'(x)}}{\sqrt{2gy(x)}}dx$$

  Minimize

  $$J(y) = \int_A^B \frac{\sqrt{1 + y'(x)}}{\sqrt{2gy(x)}}\,dx$$

  Solved by John and James Bernoulli, Newton, l'Hospital

- Find the curve enclosing largest area (Euler)

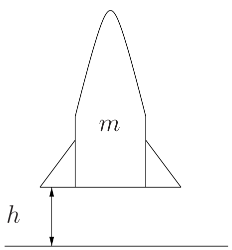
History—Optimal Control

- The space race (Sputnik, 1957)
- Pontryagin's Maximum Principle (1956)
- Bellman's Dynamic Programming (1957)
- Huge influence on engineering and other sciences:
  - Robotics—trajectory generation
  - Aeronautics—satellite orbits
  - Physics—Snell's law, conservation laws
  - Finance—portfolio theory

Goddard's Rocket Problem (1910)

How to send a rocket as high up in the air as possible?

$$\frac{d}{dt}\begin{pmatrix} v \\ h \\ m \end{pmatrix} = \begin{pmatrix} \frac{u - D}{m} - g \\ v \\ -\gamma u \end{pmatrix}$$



$(v(0), h(0), m(0)) = (0, 0, m_0)$, $g, \gamma > 0$

u motor force, $D = D(v, h)$ air resistance

Constraints: $0 \leq u \leq u_{max}$ and $m(t_f) = m_1$ (empty)

Optimization criterion: $\max_u h(t_f)$

Generalized form:

$$\min_{u:[0,t_f]\to U} \int_0^{t_f} L(x(t), u(t))\,dt + \phi(x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$
$$\psi(x(t_f)) = 0$$

Note the diffences compared to standard form:

- End time t_f is free
- Final state is constrained: $\psi(x(t_f)) = x_3(t_f) - m_1 = 0$

Solution to Goddard's Problem

Goddard's problem is on generalized form with

$$x = (v, h, m)^T, \quad L \equiv 0, \quad \phi(x) = -x_2, \quad \psi(x) = x_3 - m_1$$

$D(v, h) \equiv 0$:

- Easy: let $u(t) = u_{max}$ until $m(t) = m_1$
- Burn fuel as fast as possible, because it costs energy to lift it

$D(v, h) \not\equiv 0$:

- Hard: e.g., it can be optimal to have low speed when air resistance is high, in order to burn fuel at higher level
- Took 50 years before a complete solution was presented

General Pontryagin's Maximum Principle

Theorem: Suppose $u^* : [0, t_f] \to U$ and $x^* : [0, t_f] \to \mathbb{R}^n$ are solutions to

$$\min_{u:[0,t_f]\to U} \int_0^{t_f} L(x(t), u(t))\, dt + \phi(t_f, x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$
$$\psi(t_f, x(t_f)) = 0$$

Then, there exists $n_0 \geq 0$, $\mu \in \mathbb{R}^n$ such that $(n_0, \mu^T) \neq 0$ and

$$\min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad t \in [0, t_f]$$

where

$$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T f(x, u)$$

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}(x^*(t), u^*(t), \lambda(t), n_0)$$
$$\lambda^T(t_f) = n_0 \frac{\partial \phi}{\partial x}(t_f, x^*(t_f)) + \mu^T \frac{\partial \psi}{\partial x}(t_f, x^*(t_f))$$

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0)$$
$$= -n_0 \frac{\partial \phi}{\partial t}(t_f, x^*(t_f)) - \mu^T \frac{\partial \psi}{\partial t}(t_f, x^*(t_f))$$

Remarks:

- t_f may be a free variable
- With fixed t_f: $H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$
- ψ defines end point constraints

Example—Minimum Time Control

Bring the states of the double integrator to the origin as fast as possible

$$\min_{u:[0,t_f]\to[-1,1]} \int_0^{t_f} 1\, dt = \min_{u:[0,t_f]\to[-1,1]} t_f$$
$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t)$$
$$\psi(x(t_f)) = (x_1(t_f), x_2(t_f))^T = (0, 0)^T$$

Optimal control is the bang-bang control

$$u^*(t) = \arg \min_{u \in [-1,1]} 1 + \lambda_1(t) x_2^*(t) + \lambda_2(t) u$$
$$= \begin{cases} 1, & \lambda_2(t) < 0 \\ -1, & \lambda_2(t) \geq 0 \end{cases}$$

Adjoint equations $\dot{\lambda}_1(t) = 0, \quad \dot{\lambda}_2(t) = -\lambda_1(t)$ gives

$$\lambda_1(t) = c_1, \quad \lambda_2(t) = c_2 - c_1 t$$

With $u(t) = \zeta = \pm 1$, we have

$$x_1(t) = x_1(0) + x_2(0)t + \zeta t^2/2$$
$$x_2(t) = x_2(0) + \zeta t$$

Eliminating t gives curves

$$x_1(t) \pm x_2(t)^2/2 = \text{const}$$

These define the *switch curve*, where the optimal control switch

Reference Generation using Optimal Control

- Optimal control problem makes no distinction between open-loop control $u^*(t)$ and closed-loop control $u^*(t,x)$.
- We may use the optimal open-loop solution $u^*(t)$ as the reference value to a linear regulator, which keeps the system close to the wanted trajectory
- Efficient design method for nonlinear problems

Linear Quadratic Control

$$\min_{u:[0,\infty)\to\mathbb{R}^m} \int_0^\infty (x^T Q x + u^T R u)\, dt$$

with

$$\dot{x} = Ax + Bu$$

has optimal solution

$$u = -Lx$$

where $L = R^{-1}B^T S$ and $S > 0$ is the solution to

$$SA + A^T S + Q - SBR^{-1}B^T S = 0$$

Properties of LQ Control

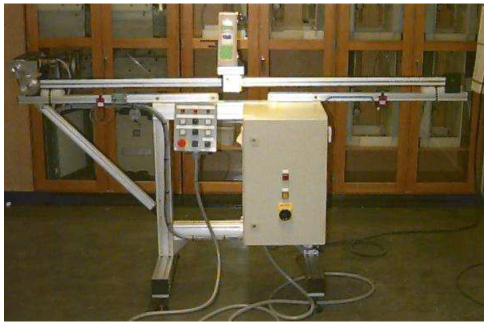
- Stabilizing
- Closed-loop system stable with $u = -\alpha(t)Lx$ for $\alpha(t) \in [1/2, \infty)$ (infinite gain margin)
- Phase margin 60 degrees

If x is not measurable, then one may use a Kalman filter; leads to linear quadratic Gaussian (LQG) control.

- But, then system may have arbitrarily poor robustness! (Doyle, 1978)

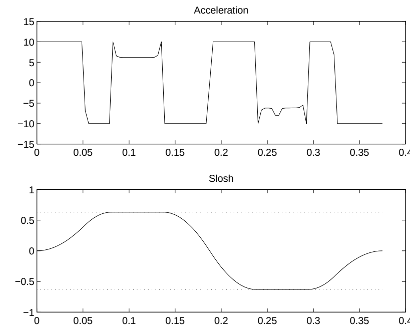
Tetra Pak Milk Race

Move milk in minimum time without spilling



[Grundelius & Bernhardsson,1999]

Given dynamics of system and maximum slosh $\phi = 0.63$, solve $\min_{u:[0,t_f]\to[-10,10]} \int_0^{t_f} 1\,dt$, where u is the acceleration.



Optimal time = 375 ms, TetraPak = 540ms

Pros & Cons for Optimal Control

+ Systematic design procedure
+ Applicable to nonlinear control problems
+ Captures limitations (as optimization constraints)
– Hard to find suitable criteria
– Hard to solve the equations that give optimal controller

SF2852 Optimal Control Theory

- **Period 3, 7.5 credits**
- Optimization and Systems Theory
  `http://www.math.kth.se/optsyst/`

Dynamic Programming: Discrete & continuous; Principle of optimality; Hamilton-Jacobi-Bellman equation

Pontryagin's Maximum principle: Main results; Special cases such as time optimal control and LQ control

Numerical Methods: Numerical solution of optimal control problems

Applications: Aeronautics, Robotics, Process Control, Bioengineering, Economics, Logistics

Today's Goal

You should be able to

- Design controllers based on optimal control theory for
  - Standard form
  - Generalized form
- Understand possibilities and limitations of optimal control

Next Lecture

Nonlinear control interpretations of

- Artificial neural networks
- Fuzzy logic and fuzzy control