EL2620 Nonlinear Control

Lecture 11

- Nonlinear controllability
- Gain scheduling

Today's Goal

You should be able to

- Determine if a nonlinear system is controllable
- Apply gain scheduling to simple examples

Controllability

Definition:

$$\dot{x} = f(x, u)$$

is **controllable** if for any x^0, x^1 there exists $T > 0$ and $u : [0, T] \to \mathbb{R}$ such that $x(0) = x^0$ and $x(T) = x^1$.

Linear Systems

Lemma:

$$\dot{x} = Ax + Bu$$

is controllable if and only if

$$W_n = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

has full rank.

Is there a corresponding result for nonlinear systems?

Controllable Linearization

Lemma: Let

$$\dot{z} = Az + Bu$$

be the linearization of

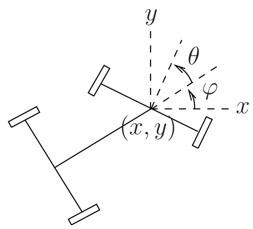
$$\dot{x} = f(x) + g(x)u$$

at $x = 0$ with $f(0) = 0$. If the linear system is controllable then the nonlinear system is controllable in a neighborhood of the origin.

Remark:

- Hence, if $\operatorname{rank} W_n = n$ then there is an $\epsilon > 0$ such that for every $x_1 \in B_\epsilon(0)$ there exists $u : [0, T] \to \mathbb{R}$ so that $x(T) = x_1$
- A nonlinear system can be controllable, even if the linearized system is not controllable

Car Example



Input: u_1 steering wheel velocity, u_2 forward velocity

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} \cos(\varphi + \theta) \\ \sin(\varphi + \theta) \\ \sin(\theta) \\ 0 \end{pmatrix} u_2 = g_1(z)u_1 + g_2(z)u_2$$

Linearization for $u_1 = u_2 = 0$ gives

$$\dot{z} = Az + B_1 u_1 + B_2 u_2$$

with $A = 0$ and

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \cos(\varphi_0 + \theta_0) \\ \sin(\varphi_0 + \theta_0) \\ \sin(\theta_0) \\ 0 \end{pmatrix}$$

$\operatorname{rank} W_n = \operatorname{rank}\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = 2 < 4$, so the linearization is not controllable. Still the car is controllable!

Linearization does not capture the controllability good enough

Lie Brackets

Lie bracket between vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Example:

$$f = \begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix}, \qquad g = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$$

$$[f, g] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix} - \begin{pmatrix} 0 & -\sin x_2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos x_2 + \sin x_2 \\ -x_1 \end{pmatrix}$$

Lie Bracket Direction

For the system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

the control

$$(u_1, u_2) = \begin{cases} (1,0), & t \in [0, \epsilon) \\ (0,1), & t \in [\epsilon, 2\epsilon) \\ (-1,0), & t \in [2\epsilon, 3\epsilon) \\ (0,-1), & t \in [3\epsilon, 4\epsilon) \end{cases}$$

gives motion

$$x(4\epsilon) = x(0) + \epsilon^2 [g_1, g_2] + O(\epsilon^3)$$

The system can move in the $[g_1, g_2]$ direction!

Proof

1. For $t \in [0, \epsilon]$, assuming ϵ small and $x(0) = x_0$, Taylor series yields

$$x(\epsilon) = x_0 + g_1(x_0)\epsilon + \frac{1}{2}\frac{dg_1}{dx} g_1(x_0)\epsilon^2 + \mathcal{O}(\epsilon^3) \qquad (1)$$

2. Similarly, for $t \in [\epsilon, 2\epsilon]$

$$x(2\epsilon) = x(\epsilon) + g_2(x(\epsilon))\epsilon + \frac{1}{2}\frac{dg_2}{dx} g_2(x(\epsilon))\epsilon^2$$

and with $x(\epsilon)$ from (1), and $g_2(x(\epsilon)) = g_2(x_0) + \frac{dg_2}{dx}\epsilon g_1(x_0)$

$$x(2\epsilon) = x_0 + \epsilon(g_1(x_0) + g_2(x_0)) + \epsilon^2 \left(\frac{1}{2}\frac{dg_1}{dx}(x_0) g_1(x_0) + \frac{dg_2}{dx}(x_0) g_1(x_0) + \frac{1}{2}\frac{dg_2}{dx}(x_0) g_2(x_0) \right)$$

Proof, continued

3. Similarly, for $t \in [2\epsilon, 3\epsilon]$

$$x(3\epsilon) = x_0 + \epsilon g_2 + \epsilon^2 \left(\frac{dg_2}{dx} g_1 - \frac{dg_1}{dx} g_2 + \frac{1}{2}\frac{dg_2}{dx} g_2 \right)$$

4. Finally, for $t \in [3\epsilon, 4\epsilon]$

$$x(4\epsilon) = x_0 + \epsilon^2 \left(\frac{dg_2}{dx} g_1 - \frac{dg_1}{dx} g_2 \right)$$

Car Example (Cont'd)

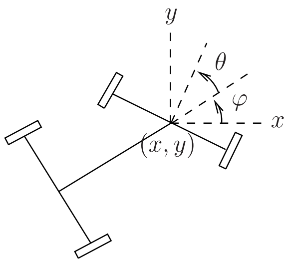
$$g_3 := [g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2$$

$$= \begin{pmatrix} 0 & 0 & -\sin(\varphi+\theta) & -\sin(\varphi+\theta) \\ 0 & 0 & \cos(\varphi+\theta) & \cos(\varphi+\theta) \\ 0 & 0 & 0 & \cos(\theta) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 0$$

$$= \begin{pmatrix} -\sin(\varphi+\theta) \\ \cos(\varphi+\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}$$

We can hence move the car in the g_3 direction ("wriggle") by applying the control sequence

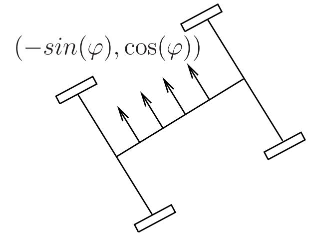
$$(u_1, u_2) = \{(1,0), (0,1), (-1,0), (0,-1)\}$$



The car can also move in the direction

$$g_4 := [g_3, g_2] = \frac{\partial g_2}{\partial x} g_3 - \frac{\partial g_3}{\partial x} g_2 = \ldots = \begin{pmatrix} -\sin(\varphi + 2\theta) \\ \cos(\varphi + 2\theta) \\ 0 \\ 0 \end{pmatrix}$$

g_4 direction corresponds to sideways movement



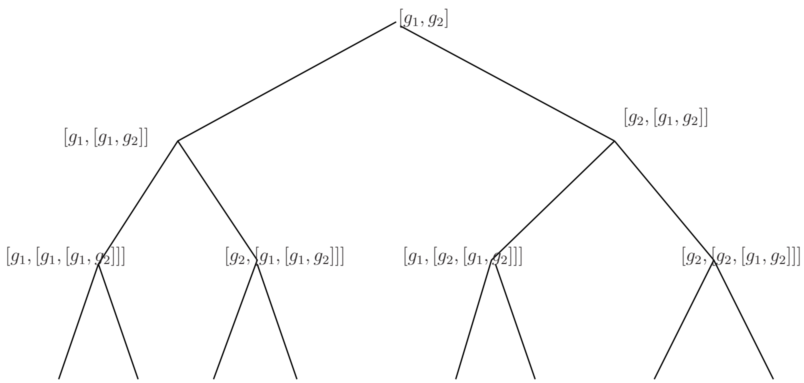
Parking Theorem

You can get out of any parking lot that is $\epsilon > 0$ bigger than your car by applying control corresponding to g_4, that is, by applying the control sequence

Wriggle, Drive, $-$Wriggle, $-$Drive

2 minute exercise: *What does the direction* $[g_1, g_2]$ *correspond to for a linear system* $\dot{x} = g_1(x)u_1 + g_2(x)u_2 = B_1 u_1 + B_2 u_2$*?*

The Lie Bracket Tree



Controllability Theorem

Theorem: The system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

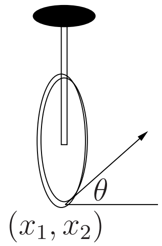
is controllable if the Lie bracket tree (together with g_1 and g_2) spans $\mathbb{R}^n$ for all x

Remark:

- The system can be steered in any direction of the Lie bracket tree

Example—Unicycle

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2$$



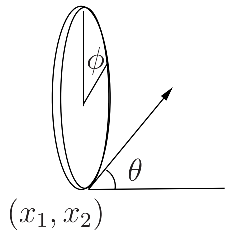
$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad [g_1, g_2] = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

Controllable because $\{g_1, g_2, [g_1, g_2]\}$ spans $\mathbb{R}^3$

2 minute exercise:

- *Show that* $\{g_1, g_2, [g_1, g_2]\}$ *spans* $\mathbb{R}^3$ *for the unicycle*
- *Is the linearization of the unicycle controllable?*

Example—Rolling Penny



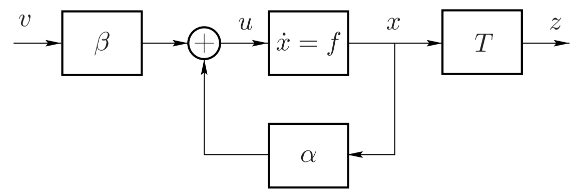
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_2$$

Controllable because $\{g_1, g_2, [g_1, g_2], [g_2, [g_1, g_2]]\}$ spans $\mathbb{R}^4$

When is Feedback Linearization Possible?

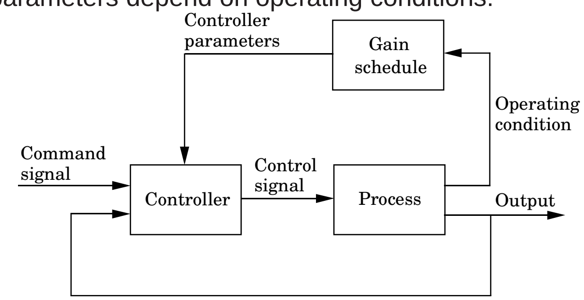
Q: When can we transform $\dot{x} = f(x) + g(x)u$ into $\dot{z} = Az + bv$ by means of feedback $u = \alpha(x) + \beta(x)v$ and change of variables $z = T(x)$ (see previous lecture)?

A: The answer requires Lie brackets and further concepts from differential geometry (see Khalil and PhD course)



Gain Scheduling

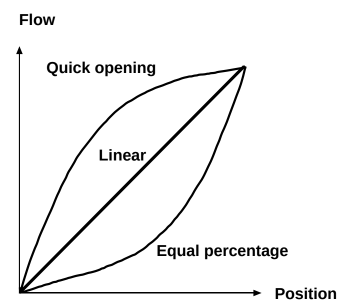
Control parameters depend on operating conditions:



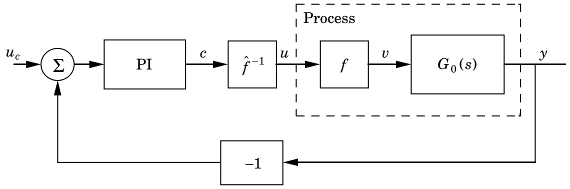
Example: PID controller with $K = K(\alpha)$, where α is the scheduling variable.

Examples of scheduling variable are production rate, machine speed, Mach number, flow rate

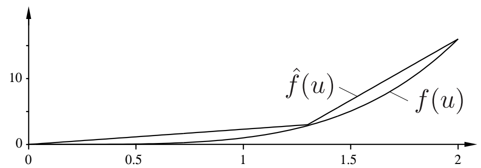
Valve Characteristics



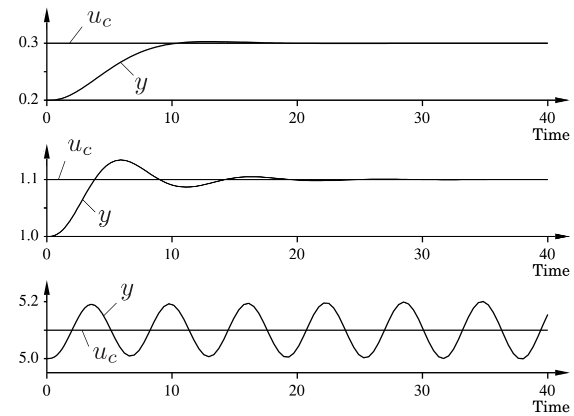
Nonlinear Valve



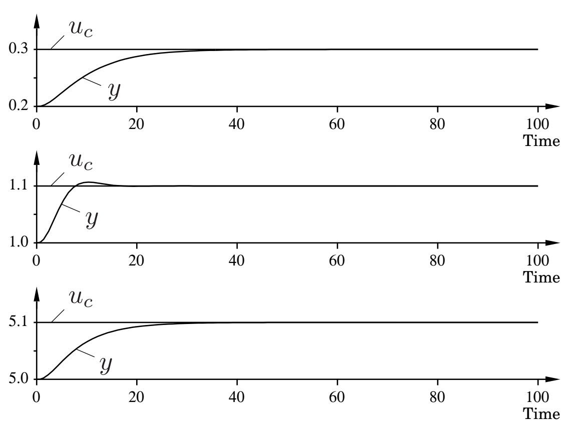
Valve characteristics



Without gain scheduling:

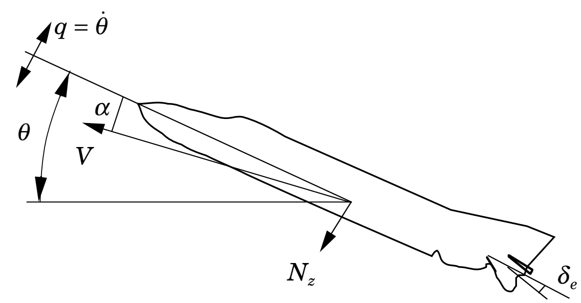


With gain scheduling:



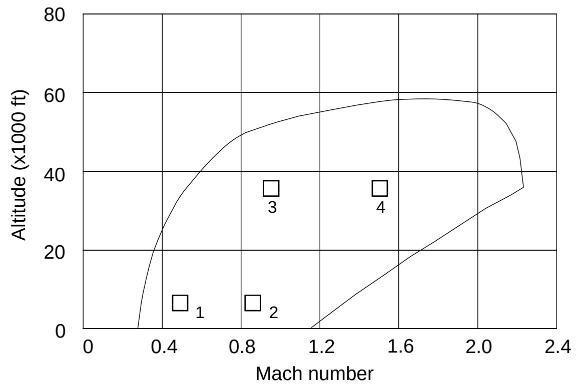
Flight Control

Pitch dynamics

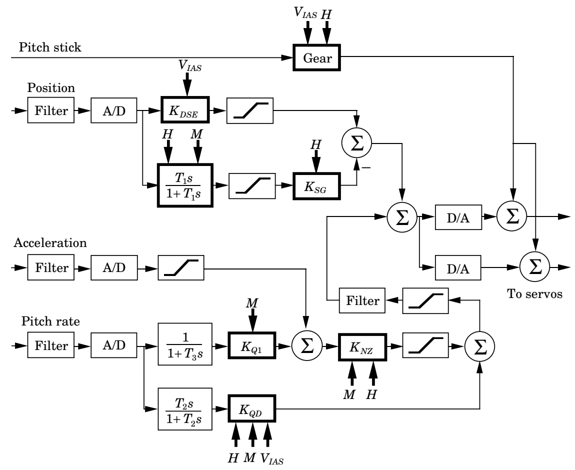


Flight Control

Operating conditions:



The Pitch Control Channel



Today’s Goal

You should be able to

- Determine if a nonlinear system is controllable
- Apply gain scheduling to simple examples

Next Lecture

- Optimal control