EL2620 Nonlinear Control

Lecture 10

- Exact feedback linearization
- Input-output linearization
- Lyapunov-based control design methods

Output Feedback and State Feedback

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

- **Output feedback:** Find $u = k(y)$ such that the closed-loop system

$$\dot{x} = f\big(x, k\big(h(x)\big)\big)$$

has nice properties.

- **State feedback:** Find $u = \ell(x)$ such that

$$\dot{x} = f(x, \ell(x))$$

has nice properties.

k and ℓ may include dynamics.

Nonlinear Controllers

- Nonlinear dynamical controller: $\dot{z} = a(z, y)$, $u = c(z)$
- Linear dynamics, static nonlinearity: $\dot{z} = Az + By$, $u = c(z)$
- Linear controller: $\dot{z} = Az + By$, $u = Cz$

Nonlinear Observers

What if x is not measurable?

$$\dot{x} = f(x, u), \quad y = h(x)$$

Simplest observer

$$\dot{\widehat{x}} = f(\widehat{x}, u)$$

Feedback correction, as in linear case,

$$\dot{\widehat{x}} = f(\widehat{x}, u) + K(y - h(\widehat{x}))$$

Choices of K

- Linearize f at x_0, find K for the linearization
- Linearize f at $\widehat{x}(t)$, find $K = K(\widehat{x})$ for the linearization

Second case is called *Extended Kalman Filter*

Some state feedback control approaches

- Exact feedback linearization
- Input-output linearization
- Lyapunov-based design - *backstepping control*

Exact Feedback Linearization

Consider the nonlinear *control-affine* system

$$\dot{x} = f(x) + g(x)u$$

idea: use a state-feedback controller $u(x)$ to make the system linear

Example 1:

$$\dot{x} = \cos x - x^3 + u$$

The state-feedback controller

$$u(x) = -\cos x + x^3 - kx + v$$

yields the linear system

$$\dot{x} = -kx + v$$

Another Example

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$

How do we cancel the term $\sin x_2$?

Perform transformation of states into linearizable form:

$$z_1 = x_1, \quad z_2 = \dot{x}_1 = a \sin x_2$$

yields

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = a \cos x_2(-x_1^2 + u)$$

and the linearizing control becomes

$$u(x) = x_1^2 + \frac{v}{a \cos x_2}, \quad x_2 \in [-\pi/2, \pi/2]$$

Diffeomorphisms

A nonlinear state transformation $z = T(x)$ with

- T invertible for x in the domain of interest
- T and T^{-1} continuously differentiable

is called a *diffeomorphism*

Definition: A nonlinear system

$$\dot{x} = f(x) + g(x)u$$

is **feedback linearizable** if there exist a diffeomorphism T whose domain contains the origin and transforms the system into the form

$$\dot{x} = Ax + B\gamma(x)\,(u - \alpha(x))$$

with (A, B) controllable and $\gamma(x)$ nonsingular for all x in the domain of interest.

Exact vs. Input-Output Linearization

The example again, but now with an output

$$\dot{x}_1 = a\sin x_2, \quad \dot{x}_2 = -x_1^2 + u, \quad y = x_2$$

- The control law $u = x_1^2 + v/a\cos x_2$ yields

$$\dot{z}_1 = z_2\,; \quad \dot{z}_2 = v\,; \quad y = \sin^{-1}(z_2/a)$$

  which is nonlinear in the output.
- If we want a linear input-output relationship we could instead use

$$u = x_1^2 + v$$

  to obtain

$$\dot{x}_1 = a\sin x_2, \quad \dot{x}_2 = v, \quad y = x_2$$

  which is linear from v to y (but what about the *unobservable state* x_1?)

Input-Output Linearization

Use state feedback $u(x)$ to make the control-affine system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

linear from the input v to the output y

- *The general idea:* differentiate the output, $y = h(x)$, p times untill the control u appears explicitly in $y^{(p)}$, and then determine u so that

$$y^{(p)} = v$$

  i.e., $G(s) = 1/s^p$

Example: controlled van der Pol equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u$$
$$y = x_1$$

Differentiate the output

$$\dot{y} = \dot{x}_1 = x_2$$
$$\ddot{y} = \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u$$

The state feedback controller

$$u = x_1 - \epsilon(1 - x_1^2)x_2 + v \quad \Rightarrow \quad \ddot{y} = v$$

Lie Derivatives

Consider the nonlinear SISO system

$$\dot{x} = f(x) + g(x)u\,;\; x \in \mathbb{R}^n, u \in \mathbb{R}^1$$
$$y = h(x), \quad y \in \mathbb{R}$$

The derivative of the output

$$\dot{y} = \frac{dh}{dx}\dot{x} = \frac{dh}{dx}\left(f(x) + g(x)u\right) \triangleq L_f h(x) + L_g h(x)u$$

where $L_f h(x)$ and $L_g h(x)$ are the *Lie derivatives* ($L_f h$ is the derivative of h along the vector field of $\dot{x} = f(x)$)

Repeated derivatives

$$L_f^k h(x) = \frac{d(L_f^{k-1}h)}{dx}f(x), \quad L_g L_f h(x) = \frac{d(L_f h)}{dx}g(x)$$

Lie derivatives and relative degree

- The relative degree p of a system is defined as the number of integrators between the input and the output (the number of times y must be differentiated for the input u to appear)
- A linear system

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

has relative degree $p = n - m$

- A nonlinear system has relative degree p if

$$L_g L_f^{i-1} h(x) = 0, i = 1, \ldots, p-1 \, ; \quad L_g L_f^{p-1} h(x) \neq 0 \quad \forall x \in D$$

Example

The controlled van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 + u \\ y &= x_1 \end{aligned}$$

Differentiating the output

$$\begin{aligned} \dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u \end{aligned}$$

Thus, the system has relative degree $p = 2$

The input-output linearizing control

Consider a nth order SISO system with relative degree p

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

Differentiating the output

$$\begin{aligned} \dot{y} &= \frac{dh}{dx}\dot{x} = L_f h(x) + \overbrace{L_g h(x) u}^{=0} \\ &\vdots \\ y^{(p)} &= L_f^p h(x) + L_g L_f^{p-1} h(x) u \end{aligned}$$

and hence the state-feedback controller

$$u = \frac{1}{L_g L_f^{p-1} h(x)} \left(-L_f^p h(x) + v \right)$$

results in the linear input-output system

$$y^{(p)} = v$$

Zero Dynamics

- Note that the order of the linearized system is p, corresponding to the relative degree of the system
- Thus, if $p < n$ then $n - p$ states are unobservable in y.
- The dynamics of the $n - p$ states not observable in the linearized dynamics of y are called the **zero dynamics**. Corresponds to the dynamics of the system when y is forced to be zero for all times.
- A system with **unstable zero dynamics** is called non-minimum phase (and should not be input-output linearized!)

van der Pol again

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u$$

- With $y = x_1$ the relative degree $p = n = 2$ and there are no zero dynamics, thus we can transform the system into $\ddot{y} = v$. *Try it yourself!*
- With $y = x_2$ the relative degree $p = 1 < n$ and the zero dynamics are given by $\dot{x}_1 = 0$, which is not asymptotically stable (but bounded)

Lyapunov-Based Control Design Methods

$$\dot{x} = f(x, u)$$

- Find stabilizing state feedback $u = u(x)$
- Verify stability through Control Lyapunov function
- Methods depend on structure of f

Here we limit discussion to **Back-stepping control design**, which require certain f discussed later.

A simple introductory example

Consider
$$\dot{x} = \cos x - x^3 + u$$
Apply the linearizing control
$$u = -\cos x + x^3 - kx$$
Choose the Lyapunov candidate $V(x) = x^2/2$
$$V(x) > 0\,, \quad \dot{V} = -kx^2 < 0$$
Thus, the system is globally asymptotically stable

But, the term x^3 in the control law may require large control moves!

The same example

$$\dot{x} = \cos x - x^3 + u$$

Now try the control law

$$u = -\cos x - kx$$

Choose the same Lyapunov candidate $V(x) = x^2/2$

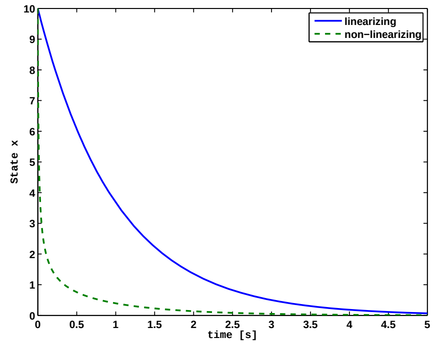
$$V(x) > 0\,, \quad \dot{V} = -x^4 - kx^2 < 0$$

Thus, also globally asymptotically stable (and more negative $\dot{V}$)

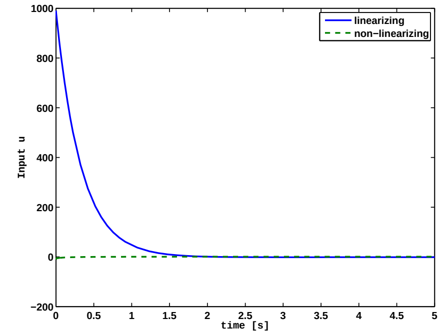
Simulating the two controllers

Simulation with $x(0) = 10$

State trajectory



Control input



The linearizing control is slower and uses excessive input

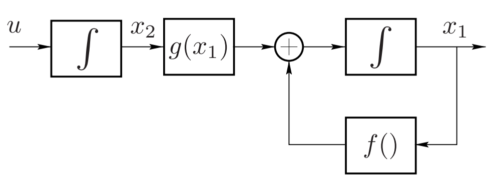
Back-Stepping Control Design

We want to design a state feedback $u = u(x)$ that stabilizes

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= u \end{aligned} \qquad (1)$$

at $x = 0$ with $f(0) = 0$.

Idea: See the system as a cascade connection. Design controller first for the inner loop and then for the outer.



Suppose the partial system

$$\dot{x}_1 = f(x_1) + g(x_1)\bar{v}$$

can be stabilized by $\bar{v} = \phi(x_1)$ and there exists Lyapunov fcn $V_1 = V_1(x_1)$ such that

$$\dot{V}_1(x_1) = \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) \le -W(x_1)$$

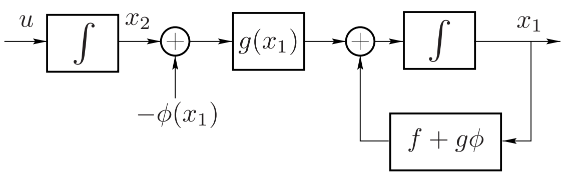
for some positive definite function W.

This is a critical assumption in backstepping control!

The Trick

Equation (1) can be rewritten as

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)[x_2 - \phi(x_1)]$$
$$\dot{x}_2 = u$$

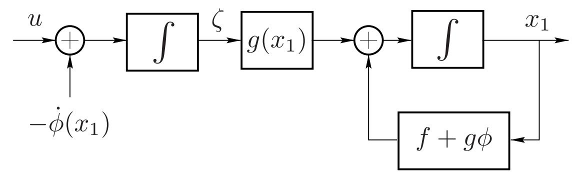


Introduce new state $\zeta = x_2 - \phi(x_1)$ and control $v = u - \dot{\phi}$:

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)\zeta$$
$$\dot{\zeta} = v$$

where

$$\dot{\phi}(x_1) = \frac{d\phi}{dx_1}\dot{x}_1 = \frac{d\phi}{dx_1}\Big(f(x_1) + g(x_1)x_2\Big)$$



Consider $V_2(x_1, x_2) = V_1(x_1) + \zeta^2/2$. Then,

$$\dot{V}_2(x_1, x_2) = \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v$$
$$\leq -W(x_1) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v$$

Choosing

$$v = -\frac{dV_1}{dx_1}g(x_1) - k\zeta, \qquad k > 0$$

gives

$$\dot{V}_2(x_1, x_2) \leq -W(x_1) - k\zeta^2$$

Hence, $x = 0$ is asymptotically stable for (1) with control law $u(x) = \dot{\phi}(x) + v(x)$.

If V_1 radially unbounded, then global stability.

Back-Stepping Lemma

Lemma: Let $z = (x_1, \ldots, x_{k-1})^T$ and

$$\dot{z} = f(z) + g(z)x_k$$
$$\dot{x}_k = u$$

Assume $\phi(0) = 0$, $f(0) = 0$,

$$\dot{z} = f(z) + g(z)\phi(z)$$

stable, and $V(z)$ a Lyapunov fcn (with $\dot{V} \leq -W$). Then,

$$u = \frac{d\phi}{dz}\Big(f(z) + g(z)x_k\Big) - \frac{dV}{dz}g(z) - (x_k - \phi(z))$$

stabilizes $x = 0$ with $V(z) + (x_k - \phi(z))^2/2$ being a Lyapunov fcn.

Strict Feedback Systems

Back-stepping Lemma can be applied to stabilize systems on strict feedback form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2\\ \dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3\\ \dot{x}_3 &= f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)x_4\\ &\vdots\\ \dot{x}_n &= f_n(x_1,\dots,x_n) + g_n(x_1,\dots,x_n)u\end{aligned}$$

where $g_k \neq 0$

Note: $x_1,\dots,x_k$ do not depend on $x_{k+2},\dots,x_n$.

2 minute exercise: *Give an example of a linear system* $\dot{x} = Ax + Bu$ *on strict feedback form.*

Back-Stepping

Back-Stepping Lemma can be applied **recursively** to a system

$$\dot{x} = f(x) + g(x)u$$

on strict feedback form.

Back-stepping generates stabilizing feedbacks $\phi_k(x_1,\dots,x_k)$ (equal to u in Back-Stepping Lemma) and Lyapunov functions

$$V_k(x_1,\dots,x_k) = V_{k-1}(x_1,\dots,x_{k-1}) + [x_k - \phi_{k-1}]^2/2$$

by “stepping back” from x_1 to u (see Khalil pp. 593–594 for details).

Back-stepping results in the final state feedback

$$u = \phi_n(x_1,\dots,x_n)$$

Example

Design back-stepping controller for

$$\dot{x}_1 = x_1^2 + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

Step 0 Verify strict feedback form
Step 1 Consider first subsystem

$$\dot{x}_1 = x_1^2 + \phi_1(x_1), \quad \dot{x}_2 = u_1$$

where $\phi_1(x_1) = -x_1^2 - x_1$ stabilizes the first equation. With $V_1(x_1) = x_1^2/2$, Back-Stepping Lemma gives

$$\begin{aligned}u_1 &= (-2x_1 - 1)(x_1^2 + x_2) - x_1 - (x_2 + x_1^2 + x_1) = \phi_2(x_1,x_2)\\ V_2 &= x_1^2/2 + (x_2 + x_1^2 + x_1)^2/2\end{aligned}$$

Step 2 Applying Back-Stepping Lemma on

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$

gives

$$u = u_2 = \frac{d\phi_2}{dz}\left(f(z) + g(z)x_n\right) - \frac{dV_2}{dz}g(z) - (x_n - \phi_2(z))$$
$$= \frac{\partial\phi_2}{\partial x_1}(x_1^2 + x_2) + \frac{\partial\phi_2}{\partial x_2}x_3 - \frac{\partial V_2}{\partial x_2} - (x_3 - \phi_2(x_1, x_2))$$

which globally stabilizes the system.

Next Lecture

- Gain scheduling
- Nonlinear controllability (*How to park a car*)