EL2620 Nonlinear Control

Lecture 9

- Nonlinear control design based on high-gain control

Today's Goal

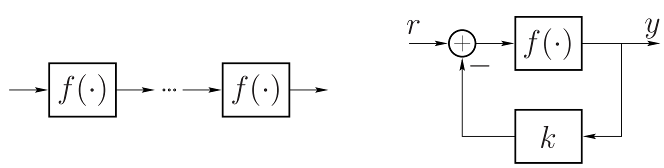
You should be able to analyze and design

- High-gain control systems
- Sliding mode controllers

History of the Feedback Amplifier

New York–San Francisco communication link 1914.

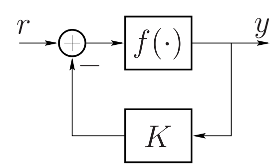
High signal amplification with low distortion was needed.

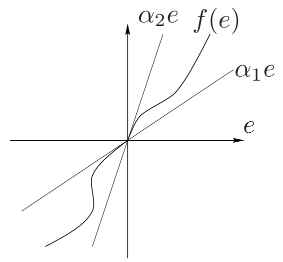


Feedback amplifiers were the solution!

Black, Bode, and Nyquist at Bell Labs 1920–1950.

Linearization Through High Gain





$$\alpha_1 \le \frac{f(e)}{e} \le \alpha_2 \quad \Rightarrow \quad \frac{\alpha_1}{1+\alpha_1 K} r \le y \le \frac{\alpha_2}{1+\alpha_2 K} r$$

choose $K \gg 1/\alpha_1$, yields

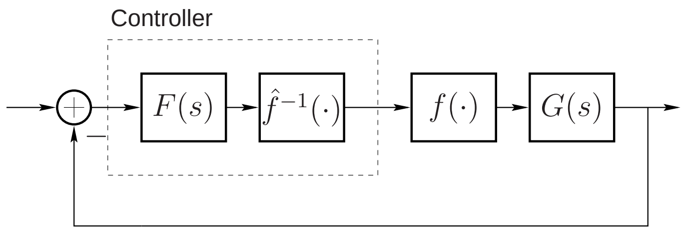
$$y \approx \frac{1}{K} r$$

A Word of Caution

Nyquist: high loop-gain may induce oscillations (due to dynamics)!

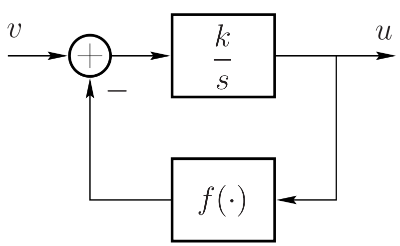
Inverting Nonlinearities

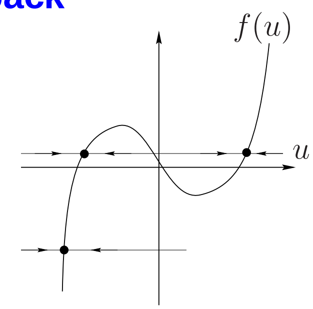
Compensation of static nonlinearity through inversion:



Should be combined with feedback as in the figure!

Remark: How to Obtain f^{-1} from f using Feedback



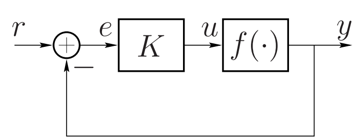


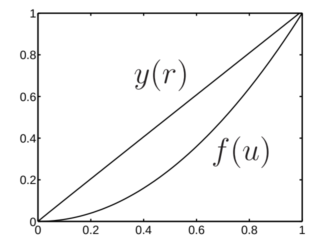
$$\dot{u} = k\big(v - f(u)\big)$$

If $k > 0$ large and $df/du > 0$, then $\dot{u} \to 0$ and

$$0 = k\big(v - f(u)\big) \quad \Leftrightarrow \quad f(u) = v \quad \Leftrightarrow \quad u = f^{-1}(v)$$

Example—Linearization of Static Nonlinearity



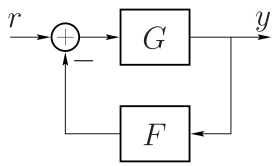


Linearization of $f(u) = u^2$ through feedback.

The case $K = 100$ is shown in the plot: $y(r) \approx r$.

The Sensitivity Function $S = (1+GF)^{-1}$

The closed-loop system is



$$G_{\text{cl}} = \frac{G}{1+GF}$$

Small perturbations dG in G gives

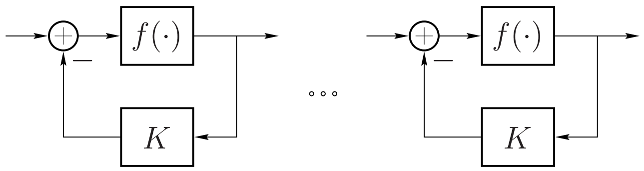
$$\frac{dG_{\text{cl}}}{dG} = \frac{1}{(1+GF)^2} \quad \Rightarrow \quad \frac{dG_{\text{cl}}}{G_{\text{cl}}} = \frac{1}{1+GF}\frac{dG}{G} = S\frac{dG}{G}$$

S is the closed-loop **sensitivity** to open-loop perturbations.

Distortion Reduction via Feedback

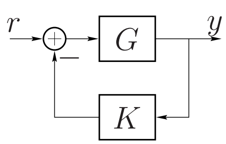
The feedback reduces distortion in each link.

Several links give distortion-free high gain.



Example—Distortion Reduction

Let $G = 1000$,
distortion $dG/G = 0.1$



Choose $K = 0.1 \quad \Rightarrow \quad S = (1+GK)^{-1} \approx 0.01$. Then

$$\frac{dG_{\text{cl}}}{G_{\text{cl}}} = S\frac{dG}{G} \approx 0.001$$

100 feedback amplifiers in series give total amplification

$$G_{\text{tot}} = (G_{\text{cl}})^{100} \approx 10^{100}$$

and total distortion

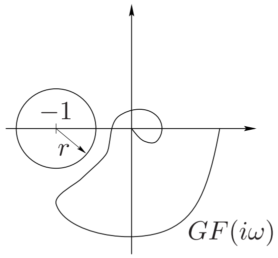
$$\frac{dG_{\text{tot}}}{G_{\text{tot}}} = (1+10^{-3})^{100} - 1 \approx 0.1$$

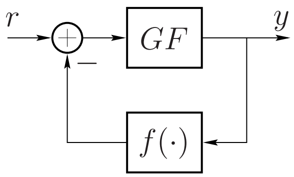
Transcontinental Communication Revolution

The feedback amplifier was patented by Black 1937.

Year	*Channels*	*Loss (dB)*	*No amp's*
1914	1	60	3–6
1923	1–4	150–400	6–20
1938	16	1000	40
1941	480	30000	600

Sensitivity and the Circle Criterion





Consider a circle $\mathcal{C} := \{z \in \mathbb{C} : |z+1| = r\}$, $r \in (0,1)$.
$GF(i\omega)$ stays outside $\mathcal{C}$ if

$$|1 + GF(i\omega)| > r \qquad \Leftrightarrow \qquad |S(i\omega)| \le r^{-1}$$

Then, the Circle Criterion gives stability if $\dfrac{1}{1+r} \le \dfrac{f(y)}{y} \le \dfrac{1}{1-r}$

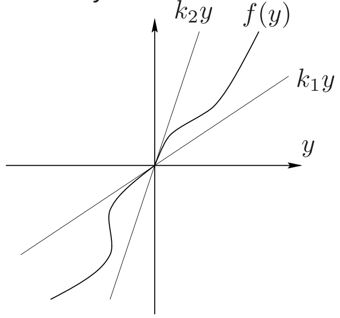
Small Sensitivity Allows Large Uncertainty

If $|S(i\omega)|$ is small, we can choose r large (close to one).

This corresponds to a large sector for $f(\cdot)$.

Hence, $|S(i\omega)|$ small implies low sensitivity to nonlinearities.

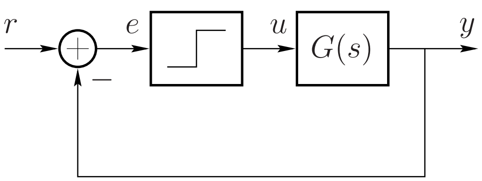
$$k_1 = \frac{1}{1+r}$$
$$k_2 = \frac{1}{1-r}$$



On–Off Control

On–off control is the simplest control strategy.

Common in temperature control, level control etc.



The relay corresponds to infinite high gain.

A Control Design Idea

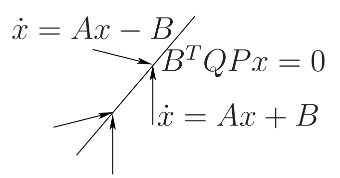
Assume $V(x) = x^T P x$, $P = P^T > 0$, represents the energy of

$$\dot{x} = Ax + Bu, \qquad u \in [-1, 1]$$

Choose u such that V decays as fast as possible:

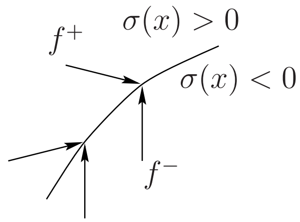
$$\dot{V} = x^T(A^T P + PA)x + 2B^T P x u$$

is minimized if $u = -\operatorname{sgn}(B^T P x)$ (Notice that $\dot{V} = a + bu$, i.e. just a segment of line in u, $-1 < u < 1$. Hence the lowest value is at an endpoint, depending on the sign of the slope b.)

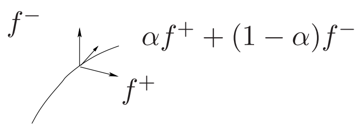


Sliding Modes

$$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$$



The **sliding mode** is $\dot{x} = \alpha f^+ + (1-\alpha) f^-$, where α satisfies $\alpha f_n^+ + (1-\alpha) f_n^- = 0$ for the normal projections of f^+, f^-



The **sliding surface** is $S = \{x : \sigma(x) = 0\}$.

Example

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u = Ax + Bu$$

$$u = -\operatorname{sgn} \sigma(x) = -\operatorname{sgn} x_2 = -\operatorname{sgn}(Cx)$$

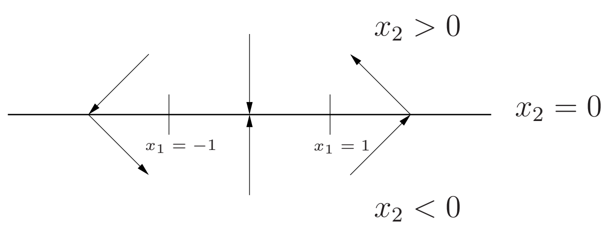
is equivalent to

$$\dot{x} = \begin{cases} Ax - B, & x_2 > 0 \\ Ax + B, & x_2 < 0 \end{cases}$$

For small x_2 we have

$$\begin{cases} \dot{x}_2(t) \approx x_1 - 1, \ \dfrac{dx_2}{dx_1} \approx 1 - x_1 & x_2 > 0 \\ \dot{x}_2(t) \approx x_1 + 1, \ \dfrac{dx_2}{dx_1} \approx 1 + x_1 & x_2 < 0 \end{cases}$$

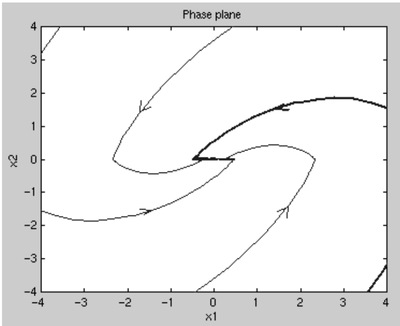
This implies the following behavior



Sliding Mode Dynamics

The dynamics along the sliding surface S is obtained by setting $u = u_{\text{eq}} \in [-1, 1]$ such that $x(t)$ stays on S.

u_{eq} is called the **equivalent control**.



Example (cont'd)

Finding $u = u_{\text{eq}}$ such that $\dot\sigma(x) = \dot x_2 = 0$ on $\sigma(x) = x_2 = 0$ gives

$$0 = \dot x_2 = x_1 - \underbrace{x_2}_{=0} + u_{\text{eq}} = x_1 + u_{\text{eq}} \quad \Rightarrow \quad u_{\text{eq}} = -x_1$$

Insert this in the equation for $\dot x_1$:

$$\dot x_1 = -\underbrace{x_2}_{=0} + u_{\text{eq}} = -x_1$$

gives the dynamics on the sliding surface $S = \{x : \; x_2 = 0\}$.

Deriving the Equivalent Control

Assume

$$\begin{aligned} \dot x &= f(x) + g(x)u \\ u &= -\operatorname{sgn}\sigma(x) \end{aligned}$$

has a stable sliding surface $S = \{x : \; \sigma(x) = 0\}$. Then, for $x \in S$,

$$0 = \dot\sigma(x) = \frac{d\sigma}{dx} \cdot \frac{dx}{dt} = \frac{d\sigma}{dx}\Big(f(x) + g(x)u\Big)$$

The equivalent control is thus given by

$$u_{\text{eq}} = -\left(\frac{d\sigma}{dx} g(x)\right)^{-1} \frac{d\sigma}{dx} f(x)$$

if the inverse exists.

Equivalent Control for Linear System

$$\begin{aligned} \dot x &= Ax + Bu \\ u &= -\operatorname{sgn}\sigma(x) = -\operatorname{sgn}(Cx) \end{aligned}$$

Assume $CB > 0$. The sliding surface $S = \{x : \; Cx = 0\}$ so

$$0 = \dot\sigma(x) = \frac{d\sigma}{dx}\Big(f(x) + g(x)u\Big) = C\big(Ax + Bu_{\text{eq}}\big)$$

gives $u_{\text{eq}} = -CAx/CB$.

Example (cont'd): For the example:

$$u_{\text{eq}} = -CAx/CB = -\begin{pmatrix}1 & -1\end{pmatrix} x = -x_1,$$

because $\sigma(x) = x_2 = 0$. (Same result as before.)

Sliding Dynamics

The dynamics on $S = \{x : \; Cx = 0\}$ is given by

$$\dot x = Ax + Bu_{\text{eq}} = \left(I - \frac{1}{CB}BC\right)Ax,$$

under the constraint $Cx = 0$, where the eigenvalues of $(I - BC/CB)A$ are equal to the zeros of $sG(s) = sC(sI - A)^{-1}B$.

Remark: The condition that $Cx = 0$ corresponds to the zero at $s = 0$, and thus this dynamic disappears on $S = \{x : \; Cx = 0\}$.

Proof

$$\dot{x} = Ax + Bu$$

$$y = Cx \;\Rightarrow\; \dot{y} = CAx + CBu \Rightarrow\; u = \frac{1}{CB}CAx - \frac{1}{CB}\dot{y} \Rightarrow$$

$$\dot{x} = \left(I - \frac{1}{CB}BC\right)Ax - \frac{1}{CB}B\dot{y}$$

Hence, the transfer function from $\dot{y}$ to u equals

$$\frac{-1}{CB} + \frac{1}{CB}CA(sI - ((I - \frac{1}{CB}BC)A))^{-1}\frac{-1}{CB}B$$

but this transfer function is also $1/(sG(s))$ Hence, the eigenvalues of $(I - BC/CB)A$ are equal to the zeros of $sG(s)$.

Design of Sliding Mode Controller

Idea: Design a control law that forces the state to $\sigma(x) = 0$. Choose $\sigma(x)$ such that the sliding mode tends to the origin. Assume

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x) + g_1(x)u \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = f(x) + g(x)u$$

Choose control law

$$u = -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)}\operatorname{sgn}\sigma(x),$$

where $\mu > 0$ is a design parameter, $\sigma(x) = p^T x$, and $p^T = \begin{pmatrix} p_1 & \dots & p_n \end{pmatrix}$ are the coefficients of a stable polynomial.

Closed-Loop Stability

Consider $V(x) = \sigma^2(x)/2$ with $\sigma(x) = p^T x$. Then,

$$\dot{V} = \sigma^T(x)\dot{\sigma}(x) = x^T p\big(p^T f(x) + p^T g(x)u\big)$$

With the chosen control law, we get

$$\dot{V} = -\mu\sigma(x)\operatorname{sgn}\sigma(x) < 0$$

so x tend to $\sigma(x) = 0$.

$$\begin{aligned} 0 = \sigma(x) &= p_1 x_1 + \cdots + p_{n-1}x_{n-1} + p_n x_n \\ &= p_1 x_n^{(n-1)} + \cdots + p_{n-1}x_n^{(1)} + p_n x_n^{(0)} \end{aligned}$$

where $x^{(k)}$ denote time derivative. Now p corresponds to a stable differential equation, and $x_n \to 0$ exponentially as $t \to \infty$. The state relations $x_{k-1} = \dot{x}_k$ now give $x \to 0$ exponentially as $t \to \infty$.

Time to Switch

Consider an initial point x_0 such that $\sigma_0 = \sigma(x_0) > 0$. Since

$$\sigma(x)\dot{\sigma}(x) = -\mu\sigma(x)\operatorname{sgn}\sigma(x)$$

it follows that as long as $\sigma(x) > 0$:

$$\dot{\sigma}(x) = -\mu$$

Hence, the time to the first switch ($\sigma(x) = 0$) is

$$t_{\text{s}} = \frac{\sigma_0}{\mu} < \infty$$

Note that $t_{\text{s}} \to 0$ as $\mu \to \infty$.

Example—Sliding Mode Controller

Design state-feedback controller for

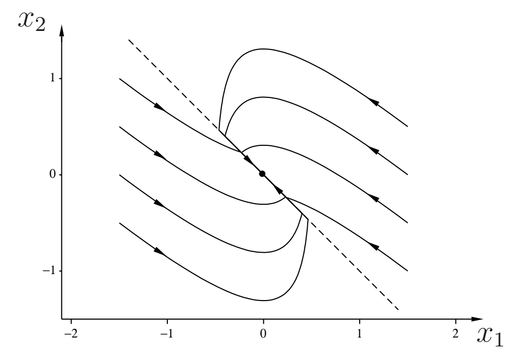
$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

Choose $p_1 s + p_2 = s + 1$ so that $\sigma(x) = x_1 + x_2$. The controller is given by

$$u = -\frac{p^T A x}{p^T B} - \frac{\mu}{p^T B} \operatorname{sgn} \sigma(x) = 2x_1 - \mu \operatorname{sgn}(x_1 + x_2)$$

Phase Portrait

Simulation with $\mu = 0.5$. Note the sliding surface $\sigma(x) = x_1 + x_2$.



Time Plots

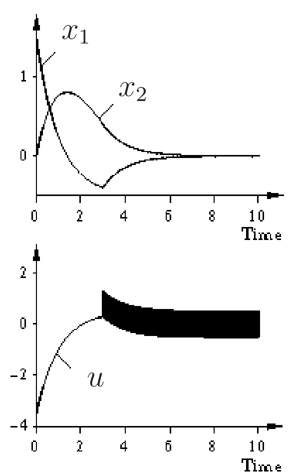
Initial condition $x(0) = \begin{pmatrix} 1.5 & 0 \end{pmatrix}^T$.

Simulation agrees well with time to switch

$$t_s = \frac{\sigma_0}{\mu} = 3$$

and sliding dynamics

$$\dot{y} = -y$$



The Sliding Mode Controller is Robust

Assume that only a model $\dot{x} = \widehat{f}(x) + \widehat{g}(x)u$ of the true system $\dot{x} = f(x) + g(x)u$ is known. Still, however,

$$\dot{V} = \sigma(x) \left[\frac{p^T (f \widehat{g}^T - \widehat{f} g^T) p}{p^T \widehat{g}} - \mu \frac{p^T g}{p^T \widehat{g}} \operatorname{sgn} \sigma(x) \right] < 0$$

if $\operatorname{sgn}(p^T g) = \operatorname{sgn}(p^T \widehat{g})$ and $\mu > 0$ is sufficiently large.

The closed-loop system is thus robust against model errors!

(High gain control with stable open loop zeros)

Comments on Sliding Mode Control

- Efficient handling of model uncertainties
- Often impossible to implement infinite fast switching
- Smooth version through low pass filter or boundary layer
- Applications in robotics and vehicle control
- Compare puls-width modulated control signals

Today's Goal

You should be able to analyze and design

- High-gain control systems
- Sliding mode controllers

Next Lecture

- Lyapunov design methods
- Exact feedback linearization