EL2620 Nonlinear Control

Lecture 6

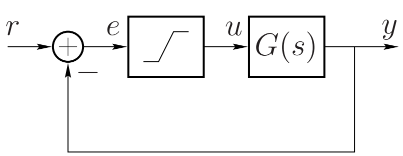
- Describing function analysis

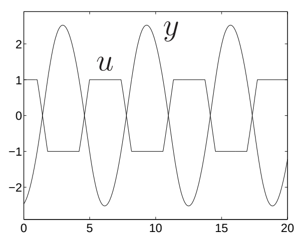
Today's Goal

You should be able to

- Derive describing functions for static nonlinearities
- Analyze existence and stability of periodic solutions by describing function analysis

Motivating Example





$G(s) = \dfrac{4}{s(s+1)^2}$ and $u =$ sat e give a stable oscillation.

- How can the oscillation be predicted?

A Frequency Response Approach

Nyquist / Bode:

A (linear) feedback system will have sustained oscillations (center) if the loop-gain is 1 at the frequency where the phase lag is -180^o

But, can we talk about the frequency response, in terms of gain and phase lag, of a static nonlinearity?

Fourier Series

A periodic function $u(t) = u(t+T)$ has a Fourier series expansion

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\sin[n\omega t + \arctan(a_n/b_n)]$$

where $\omega = 2\pi/T$ and

$$a_n(\omega) = \frac{2}{T}\int_0^T u(t)\cos n\omega t\,dt, \quad b_n(\omega) = \frac{2}{T}\int_0^T u(t)\sin n\omega t\,dt$$

Note: Sometimes we make the change of variable $t \to \phi/\omega$

The Fourier Coefficients are Optimal

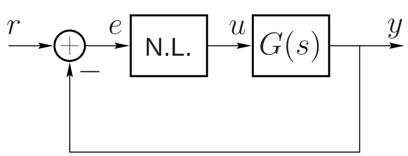
The finite expansion

$$\widehat{u}_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k}(a_n \cos n\omega t + b_n \sin n\omega t)$$

solves

$$\min_{\hat{u}} \frac{2}{T}\int_0^T \left[u(t) - \widehat{u}_k(t)\right]^2 dt$$

Key Idea



$e(t) = A\sin\omega t$ gives

$$u(t) = \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\sin[n\omega t + \arctan(a_n/b_n)]$$

If $|G(in\omega)| \ll |G(i\omega)|$ for $n \geq 2$, then $n = 1$ suffices, so that

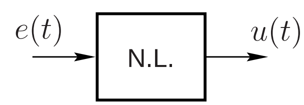
$$y(t) \approx |G(i\omega)|\sqrt{a_1^2 + b_1^2}\sin[\omega t + \arctan(a_1/b_1) + \arg G(i\omega)]$$

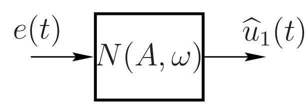
That is, we assume all higher harmonics are filtered out by G

Definition of Describing Function

The **describing function** is

$$N(A,\omega) = \frac{b_1(\omega) + ia_1(\omega)}{A}$$





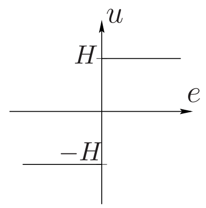
If G is low pass and $a_0 = 0$, then

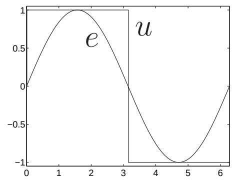
$$\widehat{u}_1(t) = |N(A,\omega)|A\sin[\omega t + \arg N(A,\omega)]$$

can be used instead of $u(t)$ to analyze the system.

Amplitude dependent gain and phase shift!

Describing Function for a Relay





$$\phi = 2\pi t/T$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi\, d\phi = \frac{2}{\pi}\int_0^{\pi} H\sin\phi\, d\phi = \frac{4H}{\pi}$$

The describing function for a relay is thus

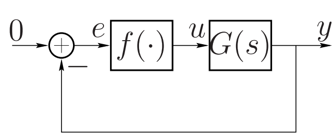
$$N(A) = \frac{b_1(\omega) + ia_1(\omega)}{A} = \frac{4H}{\pi A}$$

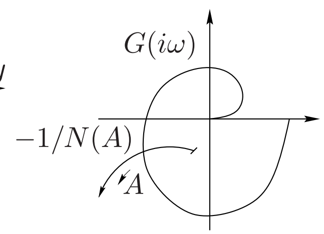
Odd Static Nonlinearities

Assume $f(\cdot)$ and $g(\cdot)$ are odd (i.e. $f(-e) = -f(e)$) static nonlinearities with describing functions N_f and N_g. Then,

- $\text{Im}\, N_f(A, \omega) = 0$
- $N_f(A, \omega) = N_f(A)$
- $N_{\alpha f}(A) = \alpha N_f(A)$
- $N_{f+g}(A) = N_f(A) + N_g(A)$

Existence of Periodic Solutions



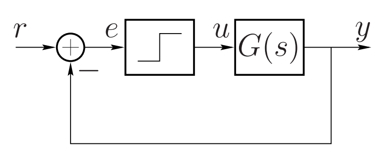


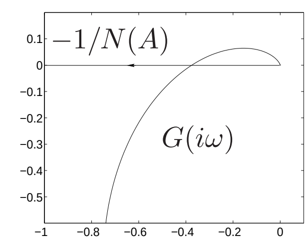
Proposal: sustained oscillations if loop-gain 1 *and phase-lag* -180^o

$$G(i\omega)N(A) = -1$$

The intersections of the curves $G(i\omega)$ and $-1/N(A)$ give ω and A for a possible periodic solution.

Periodic Solutions in Relay System



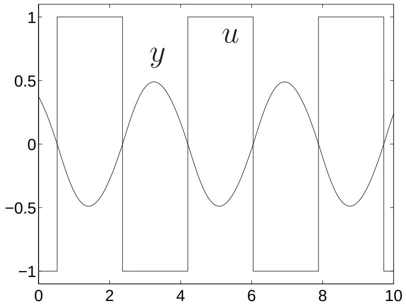


$$G(s) = \frac{3}{(s+1)^3}$$ with feedback $u = -\text{sgn}\, y$

No phase lag in $f(\cdot)$, $\arg G(i\omega) = -\pi$ for $\omega = \sqrt{3} = 1.7$

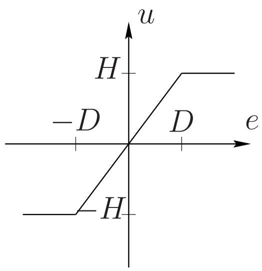
$$G(i\sqrt{3}) = -3/8 = -1/N(A) = -\pi A/4 \quad \Rightarrow \quad A = 12/8\pi \approx 0.48$$

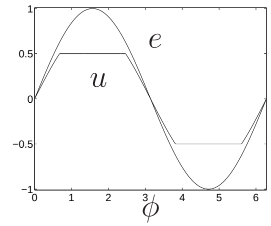
The prediction via the describing function agrees very well with the true oscillations:



Note that G filters out almost all higher-order harmonics.

Describing Function for a Saturation





Let $e(t) = A\sin\omega t = A\sin\phi$. First set $H = D$. Then for $\phi \in (0,\pi)$

$$u(\phi) = \begin{cases} A\sin\phi, & \phi \in (0,\phi_0) \cup (\pi-\phi_0,\pi) \\ D, & \phi \in (\phi_0, \pi-\phi_0) \end{cases}$$

where $\phi_0 = \arcsin D/A$.

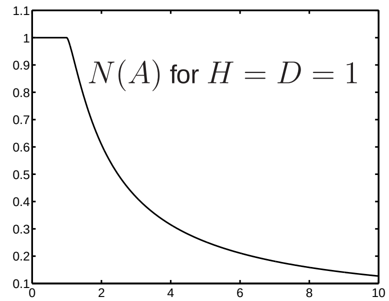
$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi = 0$$

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi\, d\phi = \frac{4}{\pi}\int_0^{\pi/2} u(\phi)\sin\phi\, d\phi \\ &= \frac{4A}{\pi}\int_0^{\phi_0}\sin^2\phi\, d\phi + \frac{4D}{\pi}\int_{\phi_0}^{\pi/2}\sin\phi\, d\phi \\ &= \frac{A}{\pi}\left(2\phi_0 + \sin 2\phi_0\right) \end{aligned}$$

Hence, if $H = D$, then $N(A) = \dfrac{1}{\pi}\left(2\phi_0 + \sin 2\phi_0\right)$.

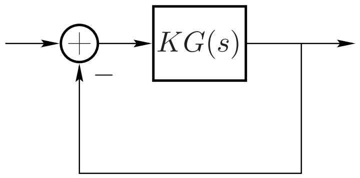
If $H \neq D$, then the rule $N_{\alpha f}(A) = \alpha N_f(A)$ gives

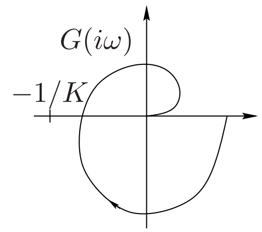
$$N(A) = \frac{H}{D\pi}\left(2\phi_0 + \sin 2\phi_0\right)$$



5 minute exercise: *What oscillation amplitude and frequency do the describing function analysis predict for the "Motivating Example"?*

The Nyquist Theorem

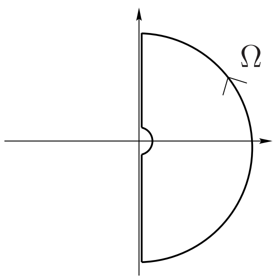


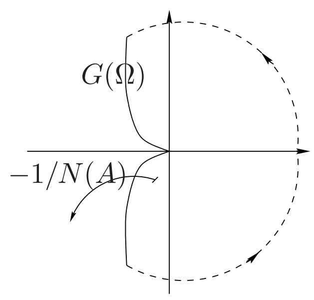


Assume that G is stable, and K is a positive gain.

- If $G(i\omega)$ goes through the point $-1/K$ the closed-loop system displays sustained oscillations
- If $G(i\omega)$ encircles the point $-1/K$, then the closed-loop system is unstable (growing amplitude oscillations).
- If $G(i\omega)$ does not encircle the point $-1/K$, then the closed-loop system is stable (damped oscillations)

Stability of Periodic Solutions

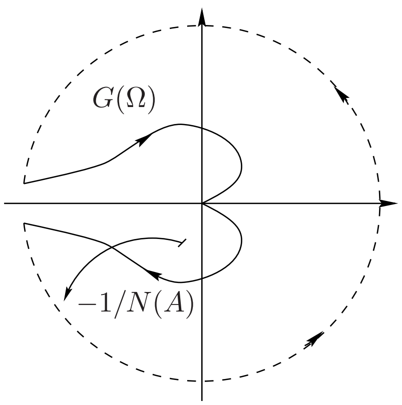




Assume that $G(s)$ is stable.

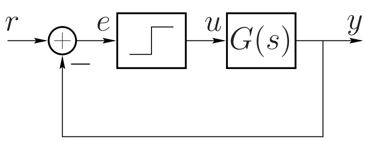
- If $G(\Omega)$ encircles the point $-1/N(A)$, then the oscillation amplitude is increasing.
- If $G(\Omega)$ does not encircle the point $-1/N(A)$, then the oscillation amplitude is decreasing.

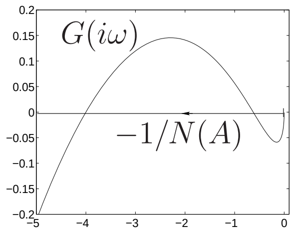
An Unstable Periodic Solution



An intersection with amplitude A_0 is unstable if $A < A_0$ leads to decreasing amplitude and $A > A_0$ leads to increasing.

Stable Periodic Solution in Relay System



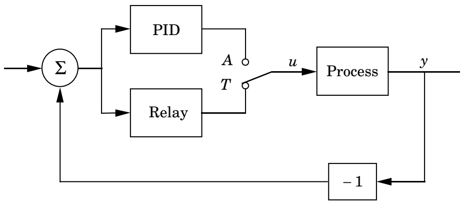


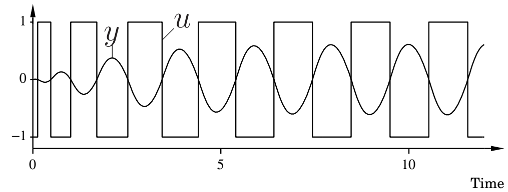
$$G(s) = \frac{(s+10)^2}{(s+1)^3} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

gives one stable and one unstable limit cycle. The left most intersection corresponds to the stable one.

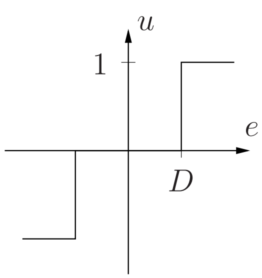
Automatic Tuning of PID Controller

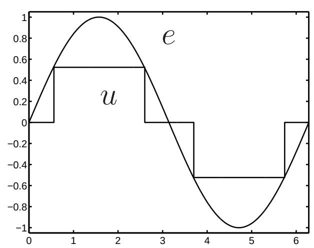
Period and amplitude of relay feedback limit cycle can be used for autotuning:





Describing Function for a Quantizer





Let $e(t) = A\sin\omega t = A\sin\phi$. Then for $\phi \in (0,\pi)$

$$u(\phi) = \begin{cases} 0, & \phi \in (0,\phi_0) \\ 1, & \phi \in (\phi_0, \pi - \phi_0) \end{cases}$$

where $\phi_0 = \arcsin D/A$.

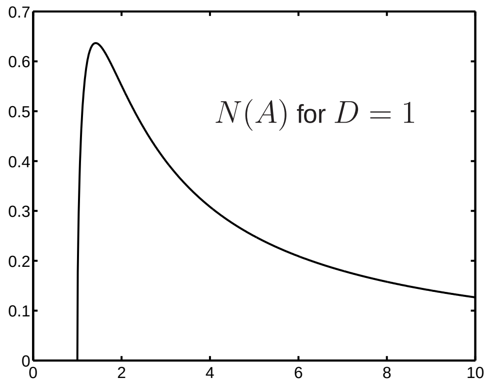
$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi d\phi = \frac{4}{\pi}\int_{\phi_0}^{\pi/2} \sin\phi d\phi$$

$$= \frac{4}{\pi}\cos\phi_0 = \frac{4}{\pi}\sqrt{1 - D^2/A^2}$$

$$N(A) = \begin{cases} 0, & A < D \\ \frac{4}{\pi A}\sqrt{1 - D^2/A^2}, & A \geq D \end{cases}$$

Plot of Describing Function Quantizer



Notice that $N(A) \approx 1.3/A$ for large amplitudes

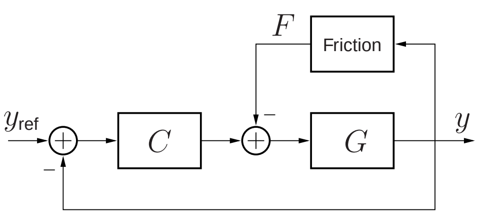
Describing Function Pitfalls

Describing function analysis can give erroneous results.

- A DF may predict a limit cycle even if one does not exist.
- A limit cycle may exist even if the DF does not predict it.
- The predicted amplitude and frequency are only approximations and can be far from the true values.

Accuracy of Describing Function Analysis

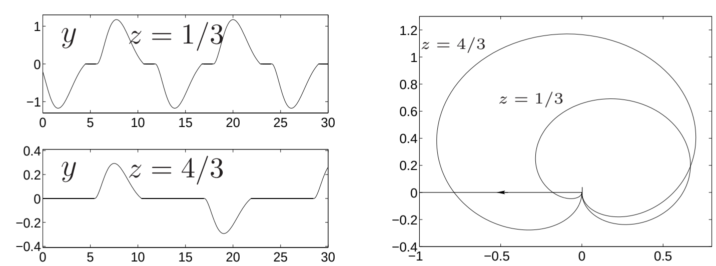
Control loop with friction $F = \text{sgn}\, y$:



Corresponds to

$$\frac{G}{1+GC} = \frac{s(s-z)}{s^3+2s^2+2s+1} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

The oscillation depends on the zero at $s = z$.

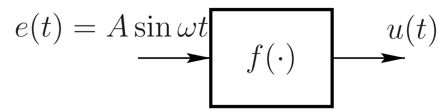


DF gives period times and amplitudes $(T, A) = (11.4, 1.00)$ and $(17.3, 0.23)$, respectively.

Accurate results only if y is close to sinusoidal!

2 minute exercise: *What is $N(A)$ for $f(x) = x^2$?*

Harmonic Balance



A few more Fourier coefficients in the truncation

$$\widehat{u}_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k} (a_n \cos n\omega t + b_n \sin n\omega t)$$

may give much better result. Describing function corresponds to $k = 1$ and $a_0 = 0$.

Example: $f(x) = x^2$ gives $u(t) = (1 - \cos 2\omega t)/2$. Hence by considering $a_0 = 1$ and $a_2 = 1/2$ we get the exact result.

Analysis of Oscillations—A Summary

Time-domain:

- Poincaré maps and Lyapunov functions
- Rigorous results but only for simple examples
- Hard to use for large problems

Frequency-domain:

- Describing function analysis
- Approximate results
- Powerful graphical methods

Today's Goal

You should be able to

- Derive describing functions for static nonlinearities
- Analyze existence and stability of periodic solutions by describing function analysis

Next Lecture

- Saturation and anti-windup compensation
- Friction modeling and compensation