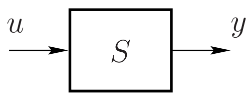
EL2620 Nonlinear Control

Lecture 5

- Input–output stability

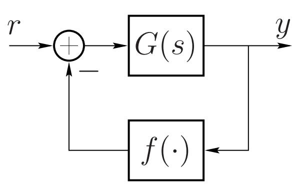


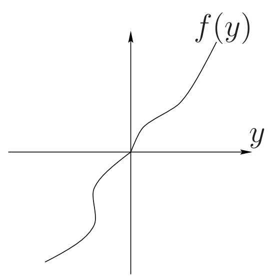
Today's Goal

You should be able to

- derive the gain of a system
- analyze stability using
  - Small Gain Theorem
  - Circle Criterion
  - Passivity

History



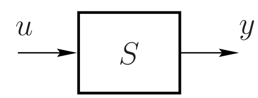


For what $G(s)$ and $f(\cdot)$ is the closed-loop system stable?

- Luré and Postnikov's problem (1944)
- Aizerman's conjecture (1949) (False!)
- Kalman's conjecture (1957) (False!)
- Solution by Popov (1960) (Led to the Circle Criterion)

Gain

Idea: Generalize the concept of gain to nonlinear dynamical systems



The **gain** γ of S is **the largest amplification** from u to y

Here S can be a constant, a matrix, a linear time-invariant system, etc

Question: How should we measure the size of u and y?

Norms

A norm $\|\cdot\|$ measures size

Definition:
A norm is a function $\|\cdot\| : \Omega \to \mathbb{R}^+$, such that for all $x, y \in \Omega$

- $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha| \cdot \|x\|$, for all $\alpha \in \mathbb{R}$

Examples:

Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$

Max norm: $\|x\| = \max\{|x_1|, \ldots, |x_n|\}$

Gain of a Matrix

Every matrix $M \in \mathbf{C}^{n\times n}$ has a **singular value decomposition**

$$M = U\Sigma V^*$$

where

$$\Sigma = \text{diag}\,\{\sigma_1, \ldots, \sigma_n\}\,; \quad U^*U = I\,; \quad V^*V = I$$

σ_i - the singular values

The "gain" of M is the largest singular value of M:

$$\sigma_{\max}(M) = \sigma_1 = \sup_{x\in\mathbb{R}^n} \frac{\|Mx\|}{\|x\|}$$

where $\|\cdot\|$ is the Euclidean norm.

Eigenvalues are not gains

The spectral radius of a matrix M

$$\rho(M) = \max_i |\lambda_i(M)|$$

is **not a gain**.

Why? *What amplification is described by the eigenvalues?*

Signal Norms

A signal x is a function $x : \mathbb{R}^+ \to \mathbb{R}$.

A signal norm $\|\cdot\|_k$ is a norm on the space of signals x.

Examples:

2-norm (energy norm): $\|x\|_2 = \sqrt{\int_0^\infty |x(t)|^2 dt}$

sup-norm: $\|x\|_\infty = \sup_{t\in\mathbb{R}^+} |x(t)|$

Parseval's Theorem

$\mathcal{L}_2$ denotes the space of signals with bounded energy: $\|x\|_2 < \infty$

Theorem: If $x, y \in \mathcal{L}_2$ have the Fourier transforms

$$X(i\omega) = \int_0^\infty e^{-i\omega t} x(t) dt, \qquad Y(i\omega) = \int_0^\infty e^{-i\omega t} y(t) dt,$$

then

$$\int_0^\infty y(t)x(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty Y^*(i\omega)X(i\omega)d\omega.$$

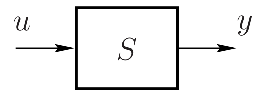
In particular,

$$\|x\|_2^2 = \int_0^\infty |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty |X(i\omega)|^2 d\omega.$$

The power calculated in the time domain equals the power calculated in the frequency domain

System Gain

A system S is a map from $\mathcal{L}_2$ to $\mathcal{L}_2$: $y = S(u)$.

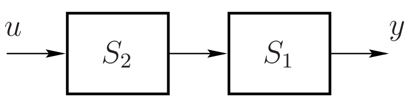


The gain of S is defined as $\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$

Example: The gain of a scalar static system $y(t) = \alpha u(t)$ is

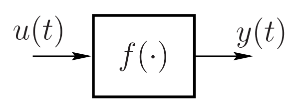
$$\gamma(\alpha) = \sup_{u \in \mathcal{L}_2} \frac{\|\alpha u\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{|\alpha|\|u\|_2}{\|u\|_2} = |\alpha|$$

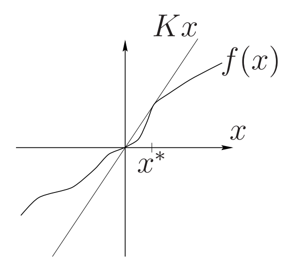
2 minute exercise: *Show that* $\gamma(S_1 S_2) \le \gamma(S_1)\gamma(S_2)$.



Gain of a Static Nonlinearity

Lemma: A static nonlinearity f such that $|f(x)| \le K|x|$ and $f(x^*) = Kx^*$ has gain $\gamma(f) = K$.



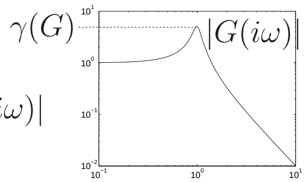


Proof: $\|y\|_2^2 = \int_0^\infty f^2\big(u(t)\big)dt \le \int_0^\infty K^2 u^2(t)dt = K^2\|u\|_2^2$,
where $u(t) = x^*$, $t \in (0,1)$, gives equality, so

$$\gamma(f) = \sup_{u \in \mathcal{L}_2} \|y\|_2 / \|u\|_2 = K$$

Gain of a Stable Linear System

Lemma:



$$\gamma(G) = \sup_{u\in\mathcal{L}_2} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega\in(0,\infty)} |G(i\omega)|$$

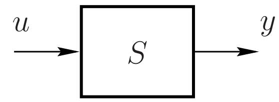
Proof: Assume $|G(i\omega)| \le K$ for $\omega \in (0,\infty)$ and $|G(i\omega^*)| = K$ for some ω^*. Parseval's theorem gives

$$\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 |U(i\omega)|^2 d\omega \le K^2 \|u\|_2^2$$

Arbitrary close to equality by choosing $u(t)$ close to $\sin\omega^* t$.

BIBO Stability

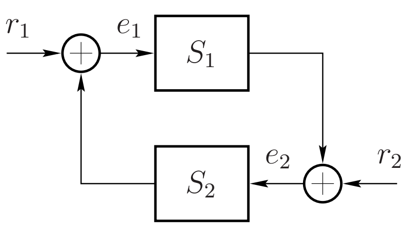
Definition:
S is bounded-input bounded-output (BIBO) stable if $\gamma(S) < \infty$.



$$\gamma(S) = \sup_{u\in\mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$$

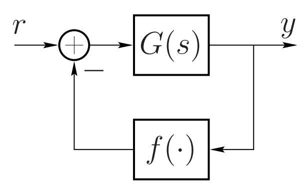
Example: If $\dot{x} = Ax$ is asymptotically stable then $G(s) = C(sI - A)^{-1}B + D$ is BIBO stable.

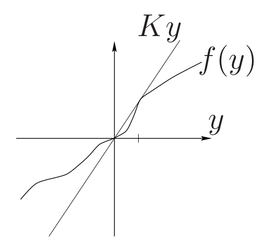
The Small Gain Theorem



Theorem: Assume S_1 and S_2 are BIBO stable. If $\gamma(S_1)\gamma(S_2) < 1$, then the closed-loop system is BIBO stable from (r_1, r_2) to (e_1, e_2)

Example—Static Nonlinear Feedback





$$G(s) = \frac{2}{(s+1)^2}, \qquad 0 \le \frac{f(y)}{y} \le K, \quad \forall y \ne 0, \quad f(0) = 0$$

$\gamma(G) = 2$ and $\gamma(f) \le K$.

Small Gain Theorem gives BIBO stability for $K \in (0, 1/2)$.

“Proof” of the Small Gain Theorem

$$\|e_1\|_2 \le \|r_1\|_2 + \gamma(S_2)[\|r_2\|_2 + \gamma(S_1)\|e_1\|_2]$$

gives

$$\|e_1\|_2 \le \frac{\|r_1\|_2 + \gamma(S_2)\|r_2\|_2}{1-\gamma(S_2)\gamma(S_1)}$$

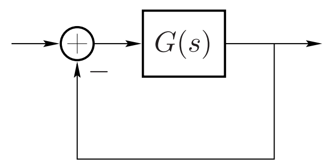
$\gamma(S_2)\gamma(S_1) < 1$, $\|r_1\|_2 < \infty$, $\|r_2\|_2 < \infty$ give $\|e_1\|_2 < \infty$.
Similarly we get

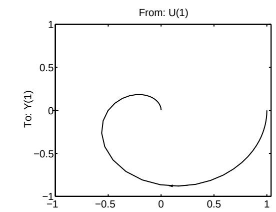
$$\|e_2\|_2 \le \frac{\|r_2\|_2 + \gamma(S_1)\|r_1\|_2}{1-\gamma(S_1)\gamma(S_2)}$$

so also e_2 is bounded.

Note: Formal proof requires $\|\cdot\|_{2e}$, see Khalil

The Nyquist Theorem

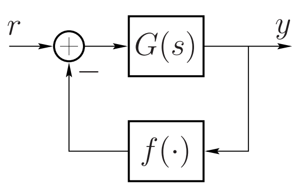


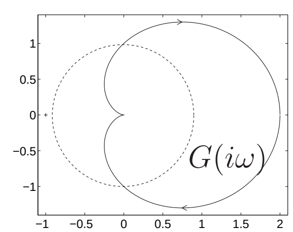


Theorem: If G has no poles in the right half plane and the Nyquist curve $G(i\omega)$, $\omega \in [0,\infty)$, does not encircle -1, then the closed-loop system is stable.

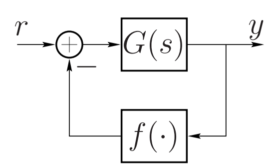
Small Gain Theorem can be Conservative

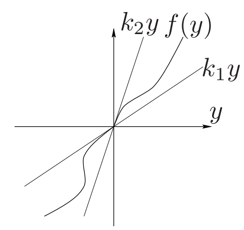
Let $f(y) = Ky$ in the previous example. Then the Nyquist Theorem proves stability for all $K \in [0,\infty)$, while the Small Gain Theorem only proves stability for $K \in (0, 1/2)$.

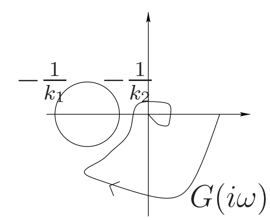




The Circle Criterion





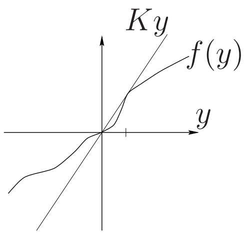


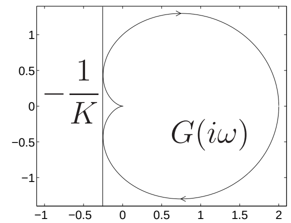
Theorem: Assume that $G(s)$ has no poles in the right half plane, and

$$0 \le k_1 \le \frac{f(y)}{y} \le k_2, \quad \forall y \ne 0, \qquad f(0) = 0$$

If the Nyquist curve of $G(s)$ does not encircle or intersect the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable.

Example—Static Nonlinear Feedback (cont'd)





The "circle" is defined by $-1/k_1 = -\infty$ and $-1/k_2 = -1/K$. Since

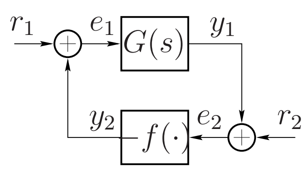
$$\min \text{Re}\, G(i\omega) = -1/4$$

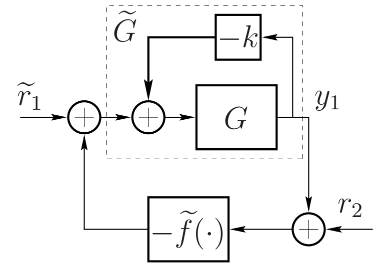
the Circle Criterion gives that the system is BIBO stable if $K \in (0, 4)$.

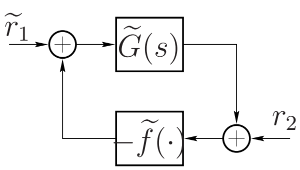
Proof of the Circle Criterion

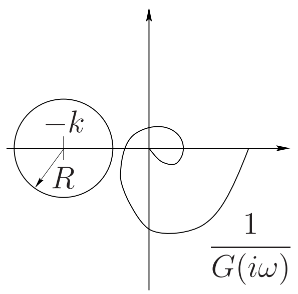
Let $k = (k_1 + k_2)/2$, $\widetilde{f}(y) = f(y) - ky$, and $\widetilde{r}_1 = r_1 - kr_2$:

$$\left|\frac{\widetilde{f}(y)}{y}\right| \le \frac{k_2 - k_1}{2} =: R, \quad \forall y \neq 0, \qquad \widetilde{f}(0) = 0$$





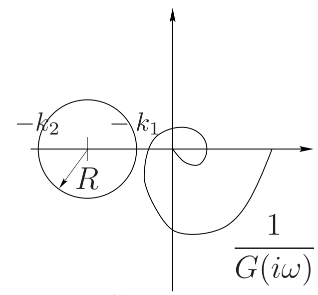


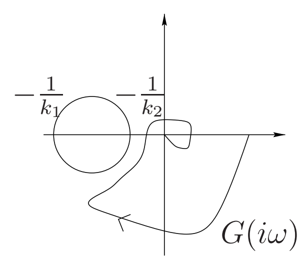


Small Gain Theorem gives stability if $|\widetilde{G}(i\omega)|R < 1$, where $\widetilde{G} = \dfrac{G}{1 + kG}$ is stable (This has to be checked later). Hence,

$$\frac{1}{|\widetilde{G}(i\omega)|} = \left|\frac{1}{G(i\omega)} + k\right| > R$$

The curve $G^{-1}(i\omega)$ and the circle $\{z \in \mathbf{C} : |z + k| > R\}$ mapped through $z \mapsto 1/z$ gives the result:





Note that $\dfrac{G}{1 + kG}$ is stable since $-1/k$ is inside the circle.

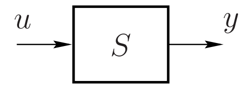
Note that $G(s)$ may have poles on the imaginary axis, e.g., integrators are allowed

Passivity and BIBO Stability

The main result: Feedback interconnections of passive systems are passive, and BIBO stable (under some additional mild criteria)

Scalar Product

Scalar product for signals y and u



$$\langle y, u \rangle_T = \int_0^T y^T(t)u(t)dt$$

If u and y are interpreted as vectors then $\langle y, u \rangle_T = |y|_T |u|_T \cos\phi$

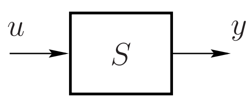
$|y|_T = \sqrt{\langle y, y \rangle_T}$ - length of y, ϕ - angle between u and y

Cauchy-Schwarz Inequality: $\langle y, u \rangle_T \leq |y|_T |u|_T$

Example: $u = \sin t$ and $y = \cos t$ are orthogonal if $T = k\pi$, because

$$\cos\phi = \frac{\langle y, u \rangle_T}{|y|_T |u|_T} = 0$$

Passive System



Definition: Consider signals $u, y : [0, T] \to \mathbb{R}^m$. The system S is **passive** if

$$\langle y, u \rangle_T \geq 0, \quad \text{for all } T > 0 \text{ and all } u$$

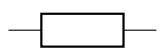
and **strictly passive** if there exists $\epsilon > 0$ such that

$$\langle y, u \rangle_T \geq \epsilon(|y|_T^2 + |u|_T^2), \quad \text{for all } T > 0 \text{ and all } u$$

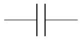
Warning: There exist many other definitions for strictly passive

2 minute exercise: Is the pure delay system $y(t) = u(t - \theta)$ passive? Consider for instance the input $u(t) = sin\left((\pi/\theta)t\right)$.

Example—Passive Electrical Components



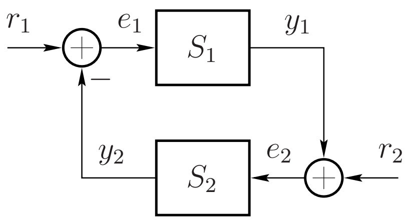
$$u(t) = Ri(t) : \langle u, i \rangle_T = \int_0^T Ri^2(t)dt \geq R\langle i, i \rangle_T \geq 0$$



$$i = C\frac{du}{dt} : \langle u, i \rangle_T = \int_0^T u(t)C\frac{du}{dt}dt = \frac{Cu^2(T)}{2} \geq 0$$

$$u = L\frac{di}{dt} : \langle u, i \rangle_T = \int_0^T L\frac{di}{dt}i(t)dt = \frac{Li^2(T)}{2} \geq 0$$

Feedback of Passive Systems is Passive



Lemma: If S_1 and S_2 are passive then the closed-loop system from (r_1, r_2) to (y_1, y_2) is also passive.

Proof:

$$\langle y, e \rangle_T = \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T = \langle y_1, r_1 - y_2 \rangle_T + \langle y_2, r_2 + y_1 \rangle_T$$
$$= \langle y_1, r_1 \rangle_T + \langle y_2, r_2 \rangle_T = \langle y, r \rangle_T$$

Hence, $\langle y, r \rangle_T \geq 0$ if $\langle y_1, e_1 \rangle_T \geq 0$ and $\langle y_2, e_2 \rangle_T \geq 0$.

Passivity of Linear Systems

Theorem: An asymptotically stable linear system $G(s)$ is **passive** if and only if

$$\text{Re}\, G(i\omega) \geq 0, \qquad \forall \omega > 0$$

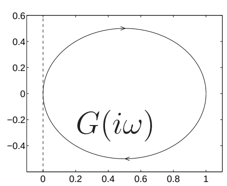
It is **strictly passive** if and only if there exists $\epsilon > 0$ such that

$$\text{Re}\, G(i\omega - \epsilon) \geq 0, \qquad \forall \omega > 0$$

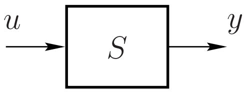
Example:

$G(s) = \dfrac{1}{s+1}$ is strictly passive,

$G(s) = \dfrac{1}{s}$ is passive but **not** strictly passive.



A Strictly Passive System Has Finite Gain



Lemma: If S is strictly passive then $\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} < \infty$.

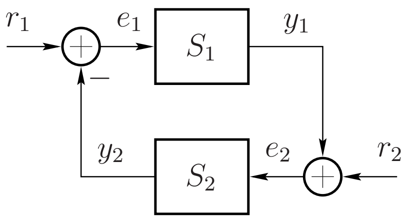
Proof:

$$\epsilon(|y|_\infty^2 + |u|_\infty^2) \leq \langle y, u \rangle_\infty \leq |y|_\infty \cdot |u|_\infty = \|y\|_2 \cdot \|u\|_2$$

Hence, $\epsilon\|y\|_2^2 \leq \|y\|_2 \cdot \|u\|_2$, so

$$\|y\|_2 \leq \frac{1}{\epsilon}\|u\|_2$$

The Passivity Theorem



Theorem: If S_1 is strictly passive and S_2 is passive, then the closed-loop system is BIBO stable from r to y.

Proof

S_1 strictly passive and S_2 passive give

$$\epsilon\left(|y_1|_T^2 + |e_1|_T^2\right) \le \langle y_1, e_1\rangle_T + \langle y_2, e_2\rangle_T = \langle y, r\rangle_T$$

Therefore

$$|y_1|_T^2 + \langle r_1 - y_2, r_1 - y_2\rangle_T \le \frac{1}{\epsilon}\langle y, r\rangle_T$$

or

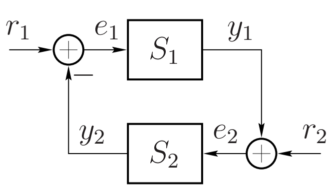
$$|y_1|_T^2 + |y_2|_T^2 - 2\langle y_2, r_2\rangle_T + |r_1|_T^2 \le \frac{1}{\epsilon}\langle y, r\rangle_T$$

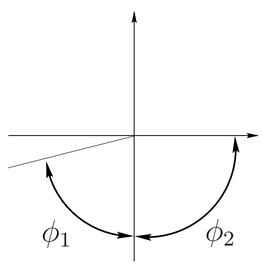
Hence

$$|y|_T^2 \le 2\langle y_2, r_2\rangle_T + \frac{1}{\epsilon}\langle y, r\rangle_T \le \left(2 + \frac{1}{\epsilon}\right)|y|_T|r|_T$$

Let $T \to \infty$ and the result follows.

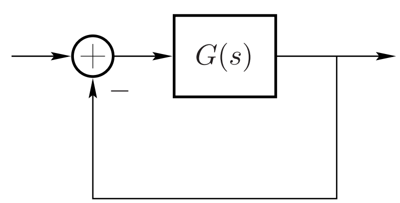
The Passivity Theorem is a "Small Phase Theorem"





S_2 passive $\Rightarrow$ $\cos\phi_2 \ge 0$ $\Rightarrow$ $|\phi_2| \le \pi/2$

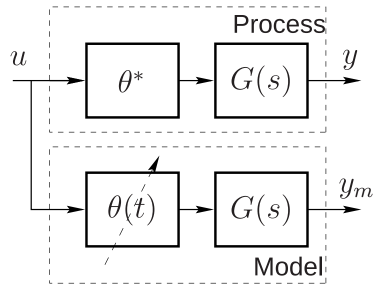
S_1 strictly passive $\Rightarrow$ $\cos\phi_1 > 0$ $\Rightarrow$ $|\phi_1| < \pi/2$



2 minute exercise: *Apply the Passivity Theorem and compare it with the Nyquist Theorem. What about conservativeness? [Compare the discussion on the Small Gain Theorem.]*

Example—Gain Adaptation

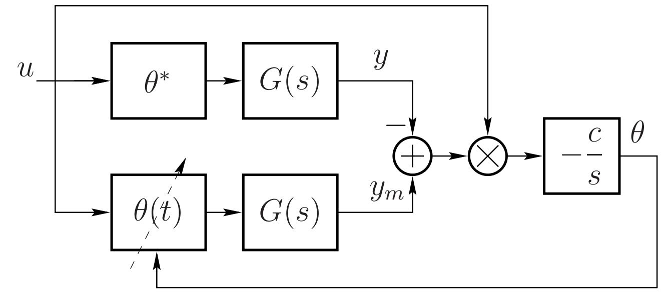
Applications in telecommunication channel estimation and in noise cancellation etc.



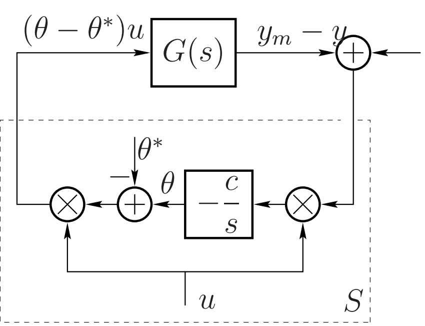
Adaptation law:

$$\frac{d\theta}{dt} = -cu(t)[y_m(t) - y(t)], \qquad c > 0.$$

Gain Adaptation—Closed-Loop System



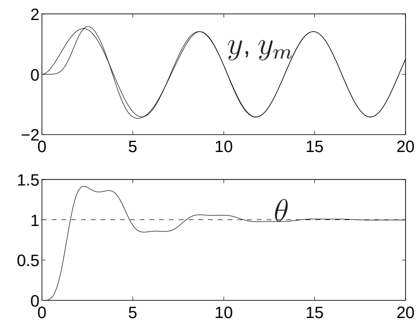
Gain Adaptation is BIBO Stable



S is **passive** (see exercises).
If $G(s)$ is **strictly passive**, the closed-loop system is BIBO stable

Simulation of Gain Adaptation

Let $G(s) = \dfrac{1}{s+1}$, $c = 1$, $u = \sin t$, and $\theta(0) = 0$.



Storage Function

Consider the nonlinear control system

$$\dot{x} = f(x,u), \qquad y = h(x)$$

A **storage function** is a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- $V(0) = 0$ and $V(x) \geq 0, \quad \forall x \neq 0$
- $\dot{V}(x) \leq u^T y, \quad \forall x, u$

Remark:

- $V(T)$ represents the stored energy in the system
- $$\underbrace{V(x(T))}_{\text{stored energy at } t=T} \leq \underbrace{\int_0^T y(t)u(t)dt}_{\text{absorbed energy}} + \underbrace{V(x(0))}_{\text{stored energy at } t=0}, \forall T > 0$$

Storage Function and Passivity

Lemma: If there exists a storage function V for a system

$$\dot{x} = f(x,u), \qquad y = h(x)$$

with $x(0) = 0$, then the system is passive.

Proof: For all $T > 0$,

$$\langle y, u\rangle_T = \int_0^T y(t)u(t)dt \geq V(x(T)) - V(x(0)) = V(x(T)) \geq 0$$

Lyapunov vs. Passivity

Storage function is a generalization of Lyapunov function

Lyapunov idea: "Energy is decreasing"

$$\dot{V} \leq 0$$

Passivity idea: "Increase in stored energy $\leq$ Added energy"

$$\dot{V} \leq u^T y$$

Example—KYP Lemma

Consider an asymptotically stable linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Assume there exists positive definite matrices P, Q such that

$$A^T P + PA = -Q, \qquad B^T P = C$$

Consider $V = 0.5x^T Px$. Then

$$\begin{aligned}\dot{V} &= 0.5(\dot{x}^T Px + x^T P\dot{x}) = 0.5x^T(A^T P + PA)x + uB^T Px \\ &= -0.5x^T Qx + uy < uy, \quad x \neq 0\end{aligned}$$

and hence the system is strictly passive. This fact is part of the Kalman-Yakubovich-Popov lemma.

Today's Goal

You should be able to

- derive the gain of a system
- analyze stability using
  - Small Gain Theorem
  - Circle Criterion
  - Passivity

Next Lecture

- Analysis of periodic solutions using describing functions