EL2620 Nonlinear Control

Lecture 4

- Lyapunov methods for stability analysis

Today's Goal

You should be able to

- Prove local and global stability of equilibria using Lyapunov's method
- Prove stability of a set (e.g., a periodic orbit) using LaSalle's invariant set theorem

Alexandr Mihailovich Lyapunov (1857–1918)

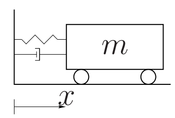
Master thesis "On the stability of ellipsoidal forms of equilibrium of rotating fluids," St. Petersburg University, 1884.

Doctoral thesis "The general problem of the stability of motion," 1892.

Formalized the idea:

If the total energy is dissipated, the system must be stable.

A Motivating Example



- Balance of forces yields

$$m\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} - \underbrace{k_0 x - k_1 x^3}_{\text{spring}}, \qquad b, k_0, k_1 > 0$$

- Total energy = kinetic + potential energy: $V = \frac{mv^2}{2} + \int_0^x F_{spring} ds$

$$V(x, \dot{x}) = m\dot{x}^2/2 + k_0 x^2/2 + k_1 x^4/4 > 0, \qquad V(0,0) = 0$$

- Change in energy along any solution $x(t)$

$$\frac{d}{dt} V(x, \dot{x}) = m\dot{x}\ddot{x} + k_0 x\dot{x} + k_1 x^3 \dot{x} = -b|\dot{x}|^3 < 0, \quad \dot{x} \neq 0$$

Stability Definitions

Recall from Lecture 3 that an equilibrium $x = 0$ of $\dot{x} = f(x)$ is

Locally stable, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

Locally asymptotically stable, if locally stable and

$$\|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

Globally asymptotically stable, if asymptotically $\forall x(0) \in \mathbb{R}^n$.

Lyapunov Stability Theorem

Theorem: Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbb{R}^n$. If there exists a $\mathbb{C}^1$ function $V : \Omega \rightarrow \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \in \Omega$, $x \neq 0$

(3) $\dot{V}(x) \leq 0$ for all $x \in \Omega$

then $x = 0$ is locally stable. Furthermore, if

(4) $\dot{V}(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

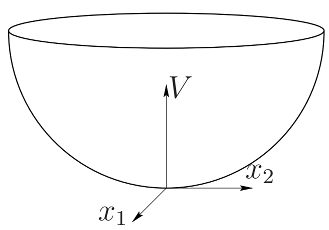
The result is called *Lyapunov's Direct Method*

Lyapunov Function

A function V that fulfills (1)–(3) is called a *Lyapunov function*.

Condition (3) means that V is non-increasing along all trajectories in Ω:

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) \leq 0$$



Conservation and Dissipation

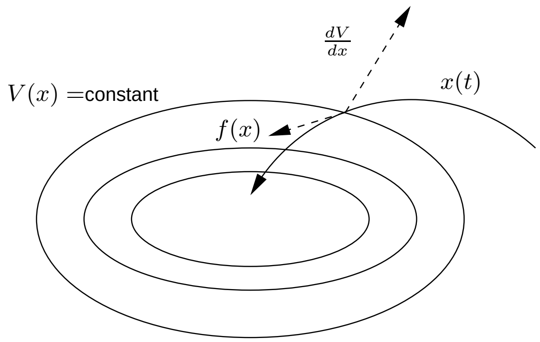
Conservation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x}f(x) = 0$, i.e. the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.

Example: Total energy of a lossless mechanical system or total fluid in a closed system.

Dissipation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x}f(x) \leq 0$, i.e. the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$ make an obtuse angle.

Example: Total energy of a mechanical system with damping or total fluid in a system that leaks.

Geometric interpretation



Vector field points into sublevel sets

Trajectories can only go to lower values of $V(x)$

Boundedness:

For an trajectory $x(t)$

$$V((x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0))$$

which means that the whole trajectory lies in the set

$$\{z \mid V(z) \leq V(x(0))\}$$

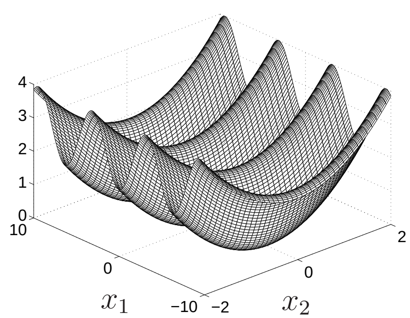
For stability it is thus important that the sublevel sets $\{z \mid V(z) \leq c)\}$ are locally bounded.

Example—Pendulum

Is the origin stable for a mathematical pendulum?

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Lyapunov function candidate: $V(x) = (1 - \cos x_1)g/\ell + x_2^2/2$



(1) $V(0) = 0$

(2) $V(x) > 0$ for $-2\pi < x_1 < 2\pi$ and $(x_1, x_2) \neq 0$

(3) $\dot{V}(x) = \frac{g}{\ell}\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = 0$, $\forall x$

Hence, $x = 0$ is locally stable.

Note that $x = 0$ is not asymptotically stable, so, of course, (4) is not fulfilled: $\dot{V}(x) \not< 0, \forall x \neq 0$.

Conservation of energy!

5 minute exercise: *Consider Example 2 from Lecture 3:*

$$\dot{x}_1 = x_2(t)$$
$$\dot{x}_2 = -x_1(t) - \epsilon x_1^2(t) x_2(t)$$

For what values of ϵ is the steady-state $(0,0)$ locally stable? Hint: try the "standard" Lyapunov function

$$V(x) = x^T x$$

Can you say something about global stability of the equilibrium?

Lyapunov Theorem for Global Asymptotic Stability

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V: R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) < 0$ for all $x \neq 0$

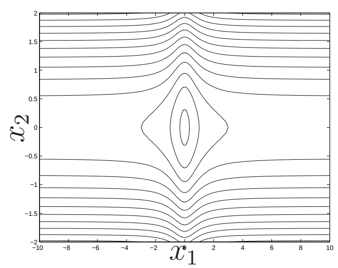
(4) $V(x) \to \infty$ as $\|x\| \to \infty$

then $x = 0$ is globally asymptotically stable.

Radial Unboundedness is Necessary

If (4) is not fulfilled, then global stability cannot be guaranteed.

Example: Assume $V(x) = x_1^2/(1+x_1^2) + x_2^2$ is a Lyapunov function for some system. Then might $x(t) \to \infty$ even if $\dot{V}(x) < 0$, as shown by the contour plot of $V(x)$:



Somewhat Stronger Assumptions

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V: R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) \leq -\alpha V(x)$ for all x

(4) The sublevel sets $\{x | V(x) \leq c\}$ are bounded for all $c \geq 0$

then $x = 0$ is globally asymptotically stable.

Proof Idea

Assume $x(t) \neq 0$ (otherwise we have $x(\tau) = 0$ for all $\tau > t$). Then

$$\frac{\dot{V}(x)}{V(x)} \leq -\alpha$$

Integrating from 0 to t gives

$$\log V(x(t)) - \log V(x(0) \leq -\alpha t \;\Rightarrow\; V(x(t)) \leq e^{-\alpha t} V(x(0))$$

Hence, $V(x(t)) \to 0$, $t \to \infty$.

Using the properties of V it follows that $x(t) \to 0$, $t \to \infty$.

Converse Lyapunov theorems

Example: If the system is globally exponentially stable

$$||x(t)|| \leq Me^{-\beta t}||x(0)||, \quad M > 0,\ \beta > 0$$

then there is a Lyapunov function that proves that it is globally asymptotically stable.

There exist also Lyapunov instability theorems!

Positive Definite Matrices

Definition: A matrix M is **positive definite** if $x^T M x > 0$ for all $x \neq 0$. It is **positive semidefinite** if $x^T M x \geq 0$ for all x.

Lemma:

- $M = M^T$ is positive definite $\iff \lambda_i(M) > 0, \forall i$
- $M = M^T$ is positive semidefinite $\iff \lambda_i(M) \geq 0, \forall i$

Note that if $M = M^T$ is positive definite, then the Lyapunov function candidate $V(x) = x^T M x$ fulfills $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$.

Symmetric Matrices

Assume that $M = M^T$. Then

$$\lambda_{\min}(M)\|x\|^2 \leq x^T M x \leq \lambda_{\max}(M)\|x\|^2$$

Hint: Use the factorization $M = U\Lambda U^T$, where U is an orthogonal matrix ($UU^T = I$) and $\Lambda =$ diag$(\lambda_1, \ldots, \lambda_n)$.

Lyapunov Stability for Linear Systems

Linear system: $\dot{x} = Ax$

Lyapunov equation: Consider the quadratic function

$$V(x) = x^T P x, \quad P = P^T > 0$$

$$\Rightarrow \quad \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T \underbrace{(PA + A^T P)}_{Q} x = -x^T Q x$$

Thus, $\dot{V} < 0 \; \forall t$ if there exist a $Q = Q^T > 0$ such that

$$PA + A^T P = -Q$$

Global Asymptotic Stability: If Q is positive definite, then the Lyapunov Stability Theorem implies global asymptotic stability, and hence the eigenvalues of A must satisfy $\operatorname{Re} \lambda_i(A) < 0$ for all i

Converse Theorem for Linear Systems

If $\operatorname{Re} \lambda_i(A) < 0$, then for every symmetric positive definite Q there exist a symmetric positive definite matrix P such that

$$PA + A^T P = -Q$$

Proof: Choose $P = \int_0^\infty e^{A^T t} Q e^{At} dt$. Then

$$\begin{aligned} A^T P + PA &= \int_0^\infty \left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A \right) dt \\ &= \int_0^\infty \left(\frac{d}{dt} e^{A^T t} Q e^{At} \right) dt = \left[e^{A^T t} Q e^{At} \right]_0^\infty = -Q \end{aligned}$$

Interpretation

Assume $\dot{x} = Ax$, $x(0) = z$. Then

$$\int_0^\infty x^T(t) Q x(t) dt = z^t \left(\int_0^\infty e^{A^T t} Q e^{At} dt \right) z = z^t P z$$

Thus $v(z) = z^T P z$ is the cost-to-go from the point z (no input) with integral quadratic cost function using weighting matrix Q.

Lyapunov's Linearization Method

Recall from Lecture 3:

Theorem: Let x_0 be an equilibrium of $\dot{x} = f(x)$ with $f \in \mathbb{C}^1$. Denote $A = \frac{\partial f}{\partial x}(x_0)$ and $\alpha(A) = \max \operatorname{Re}(\lambda(A))$.

(1) If $\alpha(A) < 0$ then x_0 is asymptotically stable

(2) If $\alpha(A) > 0$ then x_0 is unstable

Proof of (1) in Lyapunov's Linearization

Let $f(x) = Ax + g(x)$ where $\lim_{\|x\|\to 0} \|g(x)\|/\|x\| = 0$. The Lyapunov function candidate $V(x) = x^T P x$ satisfies $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, and

$$\begin{aligned}\dot{V}(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\ &= x^T(PA + A^T P)x + 2x^T P g(x) \\ &= -x^T Q x + 2x^T P g(x)\end{aligned}$$

where

$$x^T Q x \geq \lambda_{\min}(Q)\|x\|^2$$

- we need to show that $\|2x^T P g(x)\| < \lambda_{min}(Q)\|x\|^2$

For all $\gamma > 0$ there exists $r > 0$ such that

$$\|g(x)\| < \gamma\|x\|, \qquad \forall \|x\| < r$$

Thus,

$$\dot{V} < -\lambda_{min}(Q)\|x\|^2 + 2\gamma\lambda_{max}(P)\|x\|^2$$

which becomes strictly negative if we choose

$$\gamma < \frac{1}{2}\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$$

LaSalle's Theorem for Global Asymptotic Stability

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V : R^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) \leq 0$ for all x

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

(5) The only solution of $\dot{x} = f(x)$ such that $\dot{V}(x) = 0$ is $x(t) = 0$ for all t

then $x = 0$ is globally asymptotically stable.

A Motivating Example (cont'd)

$$\begin{aligned}&m\ddot{x} = -b\dot{x}|\dot{x}| - k_0 x - k_1 x^3 \\ &V(x) = (2m\dot{x}^2 + 2k_0 x^2 + k_1 x^4)/4 > 0, \qquad V(0,0) = 0 \\ &\dot{V}(x) = -b|\dot{x}|^3\end{aligned}$$

Assume that there is a trajectory with $\dot{x}(t) = 0$, $x(t) \neq 0$. Then

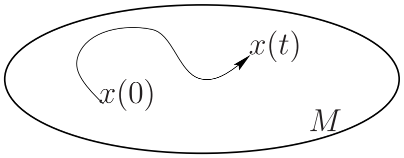
$$\frac{d}{dt}\dot{x}(t) = -\frac{k_0}{m}x(t) - \frac{k_1}{m}x^3(t) \neq 0,$$

which means that $\dot{x}(t)$ can not stay constant.

Hence, $x(t) = 0$ is the only possible trajectory for which $\dot{V}(x) = 0$, and the LaSalle theorem gives global asymptotic stability.

Invariant Sets

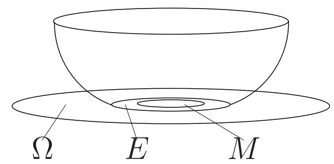
Definition: A set M is **invariant** with respect to $\dot{x} = f(x)$, if $x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

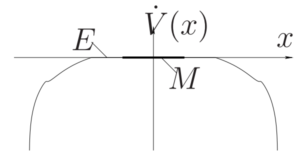


Definition: $x(t)$ **approaches** a set M as $t \to \infty$, if for each $\epsilon > 0$ there is a $T > 0$ such that $\operatorname{dist}(x(t), M) < \epsilon$ for all $t > T$. Here $\operatorname{dist}(p, M) = \inf_{x \in M} \|p - x\|$.

LaSalle's Invariant Set Theorem

Theorem: Let $\Omega \subset \mathbb{R}^n$ be a compact set invariant with respect to $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that $\dot{V}(x) \leq 0$ for $x \in \Omega$. Let E be the set of points in Ω where $\dot{V}(x) = 0$. If M is the largest invariant set in E, then every solution with $x(0) \in \Omega$ approaches M as $t \to \infty$.





Note that V does **not** have to be a positive definite function.

Example—Periodic Orbit

Show that $x(t)$ approaches $\{x : \|x\| = 1\} \cup \{0\}$ for

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

It is possible to show that $\Omega = \{\|x\| \leq R\}$ is invariant for sufficiently large $R > 0$. Let $V(x) = (x_1^2 + x_2^2 - 1)^2$.

$$\begin{aligned}\dot{V}(x) = \frac{\partial V}{\partial x} f(x) &= 2(x_1^2 + x_2^2 - 1)\frac{d}{dt}(x_1^2 + x_2^2 - 1)\\ &= -2(x_1^2 + x_2^2 - 1)^2(x_1^2 + x_2^2) \leq 0, \quad \forall x \in \Omega\end{aligned}$$

$$E = \{x \in \Omega : \dot{V}(x) = 0\} = \{x : \|x\| = 1\} \cup \{0\}$$

The largest invariant set of E is $M = E$ because

$$\frac{d}{dt}(x_1^2 + x_2^2 - 1) = -2(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2) = 0 \quad \text{for } x \in M$$