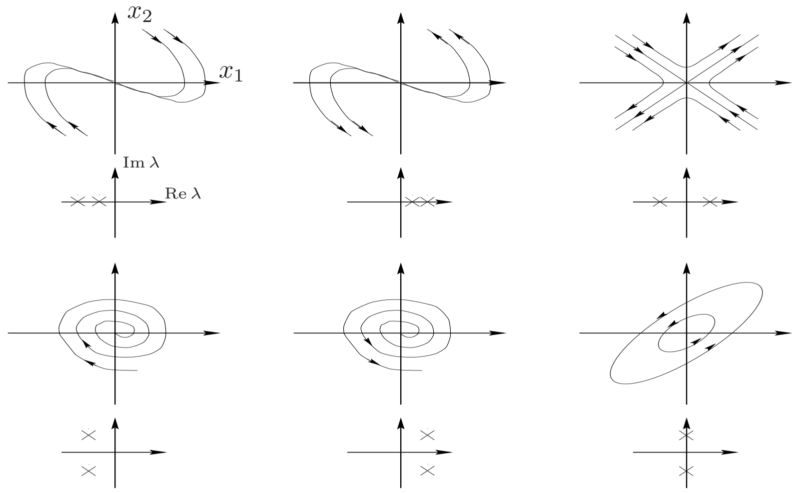
Equilibrium Points for Linear Systems



Phase-Plane Analysis for Nonlinear Systems

Close to equilibrium points “nonlinear system” $\approx$ “linear system”

Theorem: Assume

$$\dot{x} = f(x) = Ax + g(x),$$

with $\lim_{\|x\|\to 0} \|g(x)\|/\|x\| = 0$. If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

Remark: If the linearized system has a center, then the nonlinear system has either a center or a focus.

How to Draw Phase Portraits

By hand:

1. Find equilibria
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Notice that

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1}$$

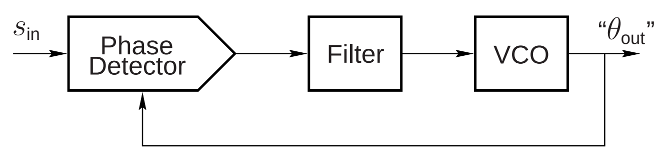
4. Try to find possible periodic orbits
5. Guess solutions

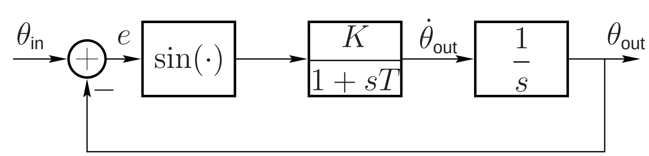
By computer:

1. Matlab: `dee` or `pplane`

Example: Phase-Locked Loop

A PLL tracks phase $\theta_{\text{in}}(t)$ of a signal $s_{\text{in}}(t) = A\sin[\omega t + \theta_{\text{in}}(t)]$.





Phase-Plane Analysis of PLL

Let $(x_1, x_2) = (\theta_{\text{out}}, \dot{\theta}_{\text{out}})$, $K, T > 0$, and $\theta_{\text{in}}(t) \equiv \theta_{\text{in}}$.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -T^{-1}x_2(t) + KT^{-1}\sin(\theta_{\text{in}} - x_1(t))$$

Equilibria are $(\theta_{\text{in}} + n\pi, 0)$ since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$
$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{in} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$

Classification of Equilibria

Linearization gives the following characteristic equations:

n even:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

$K > (4T)^{-1}$ gives stable focus
$0 < K < (4T)^{-1}$ gives stable node

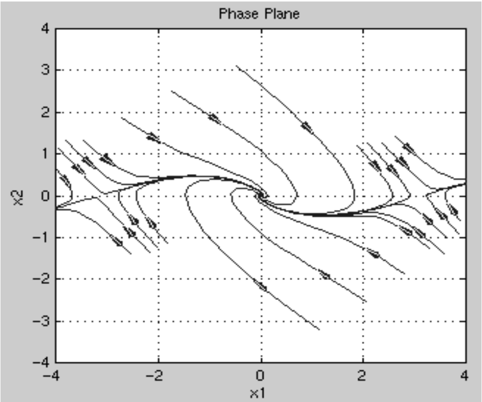
n odd:

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all $K, T > 0$

Phase-Plane for PLL

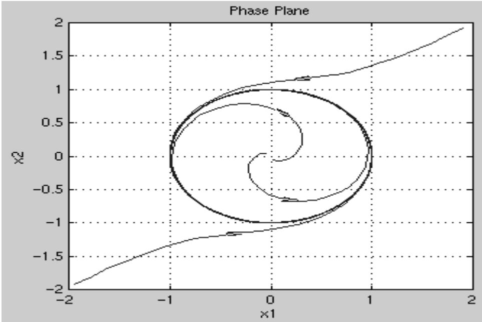
$(K, T) = (1/2, 1)$: focuses $(2k\pi, 0)$, saddle points $((2k+1)\pi, 0)$



Periodic Solutions

Example of an asymptotically stable periodic solution:

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \qquad (1)$$



Periodic solution: Polar coordinates.

$$x_1 = r\cos\theta \quad \Rightarrow \quad \dot{x}_1 = \cos\theta\dot{r} - r\sin\theta\dot{\theta}$$
$$x_2 = r\sin\theta \quad \Rightarrow \quad \dot{x}_2 = \sin\theta\dot{r} + r\cos\theta\dot{\theta}$$

implies

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now, from (1)

$$\dot{x}_1 = r(1-r^2)\cos\theta - r\sin\theta$$
$$\dot{x}_2 = r(1-r^2)\sin\theta + r\cos\theta$$

gives

$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$

Only $r = 1$ is a stable equilibrium!

A system has a **periodic solution** if for some $T > 0$

$$x(t+T) = x(t), \qquad \forall t \geq 0$$

A **periodic orbit** is the image of x in the phase portrait.

- When does there exist a periodic solution?
- When is it stable?

Note that $x(t) \equiv$ const is by convention not regarded periodic

Flow

The solution of $\dot{x} = f(x)$ is sometimes denoted

$$\phi_t(x_0)$$

to emphasize the dependence on the initial point $x_0 \in \mathbb{R}^n$

$\phi_t(\cdot)$ is called the **flow**.

Poincaré Map

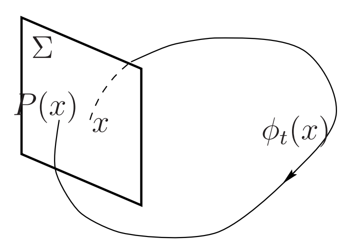
Assume $\phi_t(x_0)$ is a periodic solution with period T.

Let $\Sigma \subset \mathbb{R}^n$ be an $n-1$-dim hyperplane transverse to f at x_0.

Definition: The Poincaré map $P : \Sigma \to \Sigma$ is

$$P(x) = \phi_{\tau(x)}(x)$$

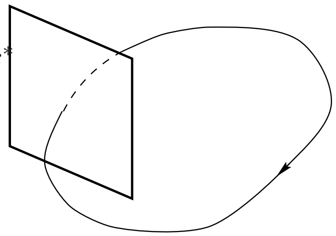
where $\tau(x)$ is the time of first return.



Existence of Periodic Orbits

A point x^* such that $P(x^*) = x^*$ corresponds to a periodic orbit.

$P(x^*) = x^*$



x^* is called a **fixed point** of P.

Stable Periodic Orbit

The linearization of P around x^* gives a matrix W such that

$$P(x) \approx Wx$$

if x is close to x^*.

- $\lambda_j(W) = 1$ for some j
- If $|\lambda_i(W)| < 1$ for all $i \neq j$, then the corresponding periodic orbit is **asymptotically stable**
- If $|\lambda_i(W)| > 1$ for some i, then the periodic orbit is **unstable**.

Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Choose $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}$.

The solution is

$$\phi_t(r_0, \theta_0) = \left([1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0 \right)$$

First return time from any point $(r_0, \theta_0) \in \Sigma$ is

$$\tau(r_0, \theta_0) = 2\pi.$$

The Poincaré map is

$$P(r_0, \theta_0) = \begin{pmatrix} [1 + (r_0^{-2} - 1)e^{-2 \cdot 2\pi}]^{-1/2} \\ \theta_0 + 2\pi \end{pmatrix}$$

$(r_0, \theta_0) = (1, 2\pi k)$ is a fixed point.

The periodic solution that corresponds to $(r(t), \theta(t)) = (1, t)$ is asymptotically stable because

$$W = \frac{dP}{d(r_0, \theta_0)}(1, 2\pi k) = \begin{pmatrix} e^{-4\pi} & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Stable periodic orbit (as we already knew for this example)

Next Lecture

- Lyapunov methods for stability analysis