# EL2620 Nonlinear Control

## Lecture 3

- Stability definitions
- Linearization
- Phase-plane analysis
- Periodic solutions





## Today's Goal

You should be able to

- Explain local and global stability
- Linearize around equilibria and trajectories
- Sketch phase portraits for two-dimensional systems
- Classify equilibria into nodes, focuses, saddle points, and center points
- Analyze stability of periodic solutions through Poincaré maps



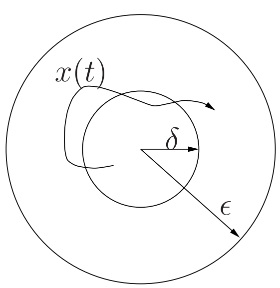


## Local Stability

Consider $\dot{x} = f(x)$ with $f(0) = 0$

**Definition:** The equilibrium $x^* = 0$ is **stable** if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0$$




If $x^* = 0$ is not stable it is called **unstable**.





## Asymptotic Stability

**Definition:** The equilibrium $x = 0$ is **asymptotically stable** if it is stable and $\delta$ can be chosen such that

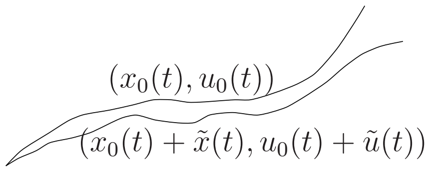
$$\|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

The equilibrium is **globally asymptotically stable** if it is stable and $\lim_{t\to\infty} x(t) = 0$ for all $x(0)$.

## Linearization Around a Trajectory

Let $(x_0(t), u_0(t))$ denote a solution to $\dot{x} = f(x, u)$ and consider another solution $(x(t), u(t)) = (x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$:

$$\begin{aligned}\dot{x}(t) &= f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t)) \\ &= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}(x_0(t), u_0(t))\tilde{x}(t) \\ &\quad + \frac{\partial f}{\partial u}(x_0(t), u_0(t))\tilde{u}(t) + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)\end{aligned}$$








Hence, for small $(\tilde{x}, \tilde{u})$, approximately

$$\dot{\tilde{x}}(t) = A(x_0(t), u_0(t))\tilde{x}(t) + B(x_0(t), u_0(t))\tilde{u}(t)$$

where

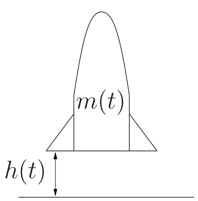
$$\begin{aligned}A(x_0(t), u_0(t)) &= \frac{\partial f}{\partial x}(x_0(t), u_0(t)) \\ B(x_0(t), u_0(t)) &= \frac{\partial f}{\partial u}(x_0(t), u_0(t))\end{aligned}$$

Note that $A$ and $B$ are time dependent. However, if $(x_0(t), u_0(t)) \equiv (x_0, u_0)$ then $A$ and $B$ are constant.





## Example




$$\begin{aligned}\dot{h}(t) &= v(t) \\ \dot{v}(t) &= -g + v_e u(t)/m(t) \\ \dot{m}(t) &= -u(t)\end{aligned}$$

Let $x_0(t) = (h_0(t), v_0(t), m_0(t))^T$, $u_0(t) \equiv u_0 > 0$, be a solution. Then,

$$\dot{\tilde{x}}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{v_e u_0}{(m_0 - u_0 t)^2} \\ 0 & 0 & 0 \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} 0 \\ \frac{v_e}{m_0 - u_0 t} \\ 1 \end{pmatrix} \tilde{u}(t)$$





## Pointwise Left Half-Plane Eigenvalues of $A(t)$ Do *Not* Impose Stability

$$A(t) = \begin{pmatrix} -1 + \alpha\cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are stable for $0 < \alpha < 2$. However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{pmatrix} x(0),$$

is unbounded solution for $\alpha > 1$.

## Lyapunov's Linearization Method

**Theorem:** Let $x_0$ be an equilibrium of $\dot{x} = f(x)$ with $f \in \mathbb{C}^1$. Denote $A = \frac{\partial f}{\partial x}(x_0)$ and $\alpha(A) = \max \mathrm{Re}(\lambda(A))$.

- If $\alpha(A) < 0$, then $x_0$ is asymptotically stable
- If $\alpha(A) > 0$, then $x_0$ is unstable

**The fundamental result for linear systems theory!**

The case $\alpha(A) = 0$ needs further investigation.

The theorem is also called *Lyapunov's Indirect Method*.

A proof is given next lecture.





## Example

The linearization of

$$\dot{x}_1 = -x_1^2 + x_1 + \sin x_2$$
$$\dot{x}_2 = \cos x_2 - x_1^3 - 5x_2$$

at the equilibrium $x_0 = (1, 0)^T$ is given by

$$A = \begin{pmatrix} -1 & 1 \\ -3 & -5 \end{pmatrix}, \qquad \lambda(A) = \{-2, -4\}$$

$x_0$ is thus an asymptotically stable equilibrium for the *nonlinear* system.





## Linear Systems Revival

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution: $x(t) = e^{At}x(0),\ t \geq 0$.

If $A$ is diagonalizable, then

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where $v_1, v_2$ are the eigenvectors of $A$ ($Av_1 = \lambda_1 v_1$ etc).

This implies that

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2,$$

where the constants $c_1$ and $c_2$ are given by the initial conditions





## Example: Two real negative eigenvalues

Given the eigenvalues $\lambda_1 < \lambda_2 < 0$, with corresponding eigenvectors $v_1$ and $v_2$, respectively.

Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

*Slow eigenvalue/vector:* $x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large $t$.
Moves along the slow eigenvector towards $x = 0$ for large $t$

*Fast eigenvalue/vector:* $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ for small $t$.
Moves along the fast eigenvector for small $t$

## Phase-Plane Analysis for Linear Systems

The location of the eigenvalues $\lambda(A)$ determines the characteristics of the trajectories.

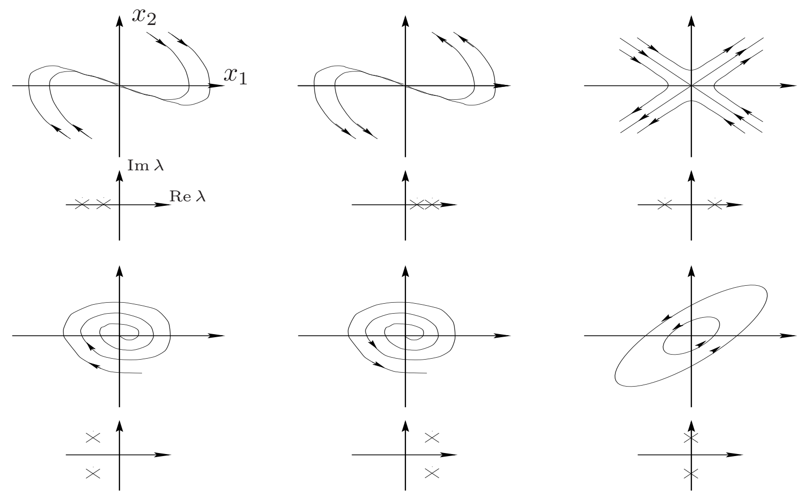
Six cases:

| | | | |
|---|---|---|---|
| | stable node | unstable node | saddle point |
| $\mathrm{Im}\,\lambda_i = 0:$ | $\lambda_1, \lambda_2 < 0$ | $\lambda_1, \lambda_2 > 0$ | $\lambda_1 < 0 < \lambda_2$ |
| $\mathrm{Im}\,\lambda_i \neq 0:$ | $\mathrm{Re}\,\lambda_i < 0$ | $\mathrm{Re}\,\lambda_i > 0$ | $\mathrm{Re}\,\lambda_i = 0$ |
| | stable focus | unstable focus | center point |





## Equilibrium Points for Linear Systems








## Example—Unstable Focus

$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \qquad \sigma, \omega > 0, \qquad \lambda_{1,2} = \sigma \pm i\omega$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t}e^{i\omega t} & 0 \\ 0 & e^{\sigma t}e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} x(0)$$

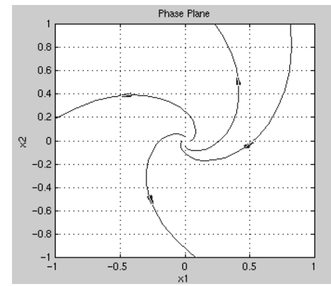
In polar coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan x_2/x_1$ ($x_1 = r\cos\theta$, $x_2 = r\sin\theta$):

$$\dot{r} = \sigma r$$
$$\dot{\theta} = \omega$$

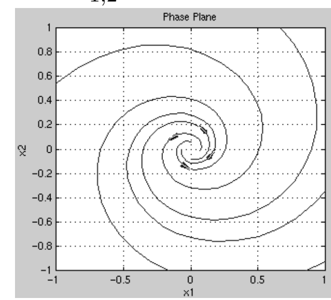




$\lambda_{1,2} = 1 \pm i$




$\lambda_{1,2} = 0.3 \pm i$



## Example—Stable Node

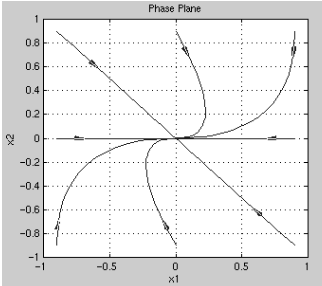
$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$(\lambda_1, \lambda_2) = (-1, -2) \quad \text{and} \quad \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$v_1$ is the slow direction and $v_2$ is the fast.








Fast: $x_2 = -x_1 + c_3$
Slow: $x_2 = 0$





**5 minute exercise:** *What is the phase portrait if* $\lambda_1 = \lambda_2$*?*

*Hint:* Two cases; only one linear independent eigenvector or all vectors are eigenvectors





## Phase-Plane Analysis for Nonlinear Systems

Close to equilibrium points “nonlinear system” $\approx$ “linear system”

**Theorem:** Assume

$$\dot{x} = f(x) = Ax + g(x),$$

with $\lim_{\|x\| \to 0} \|g(x)\|/\|x\| = 0$. If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

**Remark:** If the linearized system has a center, then the nonlinear system has either a center or a focus.

# How to Draw Phase Portraits

By hand:

1. Find equilibria
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Notice that

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_1}{\dot{x}_2}$$

4. Try to find possible periodic orbits
5. Guess solutions

By computer:

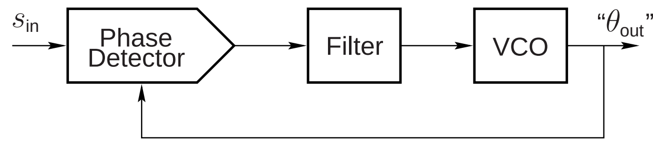
1. Matlab: `dee` or `pplane`

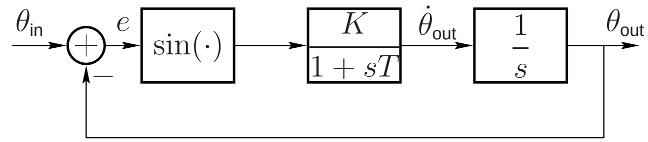




# Phase-Locked Loop

A PLL tracks phase $\theta_{\text{in}}(t)$ of a signal $s_{\text{in}}(t) = A\sin[\omega t + \theta_{\text{in}}(t)]$.








# Phase-Plane Analysis of PLL

Let $(x_1, x_2) = (\theta_{\text{out}}, \dot{\theta}_{\text{out}})$, $K, T > 0$, and $\theta_{\text{in}}(t) \equiv \theta_{\text{in}}$.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -T^{-1}x_2(t) + KT^{-1}\sin(\theta_{\text{in}} - x_1(t))$$

Equilibria are $(\theta_{\text{in}} + n\pi, 0)$ since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$
$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{in} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$





# Classification of Equilibria

Linearization gives the following characteristic equations:

$n$ even:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

$K > (4T)^{-1}$ gives stable focus
$0 < K < (4T)^{-1}$ gives stable node

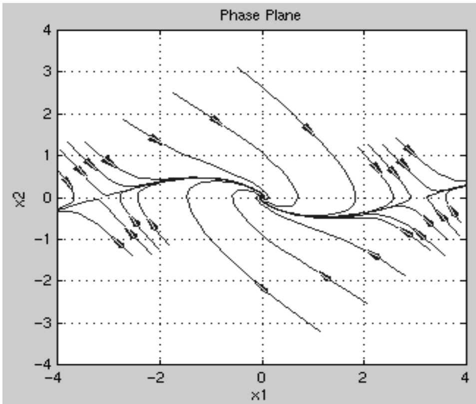
$n$ odd:

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all $K, T > 0$

## Phase-Plane for PLL

$(K,T) = (1/2, 1)$: focuses $(2k\pi, 0)$, saddle points $((2k+1)\pi, 0)$



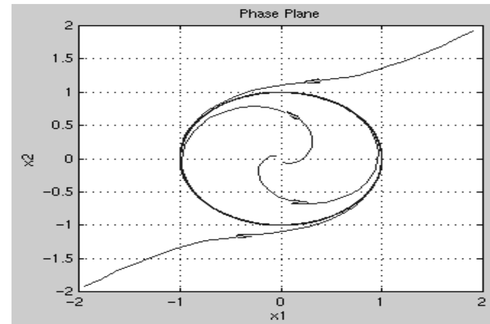





## Periodic Solutions

Example of an asymptotically stable periodic solution:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned} \qquad (1)$$








## Periodic solution: Polar coordinates.

$$x_1 = r\cos\theta \quad \Rightarrow \quad \dot{x}_1 = \cos\theta\dot{r} - r\sin\theta\dot{\theta}$$
$$x_2 = r\sin\theta \quad \Rightarrow \quad \dot{x}_2 = \sin\theta\dot{r} + r\cos\theta\dot{\theta}$$

implies

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now

$$\begin{aligned} \dot{x}_1 &= r(1-r^2)\cos\theta - r\sin\theta \\ \dot{x}_2 &= r(1-r^2)\sin\theta + r\cos\theta \end{aligned}$$

gives

$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

Only $r = 1$ is a stable equilibrium!





A system has a **periodic solution** if for some $T > 0$

$$x(t+T) = x(t), \qquad \forall t \geq 0$$

A **periodic orbit** is the image of $x$ in the phase portrait.

- When does there exist a periodic solution?
- When is it stable?

Note that $x(t) \equiv$ const is by convention not regarded periodic

# Flow

The solution of $\dot{x} = f(x)$ is sometimes denoted

$$\phi_t(x_0)$$

to emphasize the dependence on the initial point $x_0 \in \mathbb{R}^n$

$\phi_t(\cdot)$ is called the **flow**.





# Poincaré Map

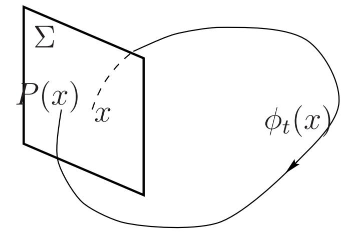
Assume $\phi_t(x_0)$ is a periodic solution with period $T$.

Let $\Sigma \subset \mathbb{R}^n$ be an $n-1$-dim hyperplane transverse to $f$ at $x_0$.

**Definition:** The Poincaré map $P : \Sigma \to \Sigma$ is

$$P(x) = \phi_{\tau(x)}(x)$$

where $\tau(x)$ is the time of first return.

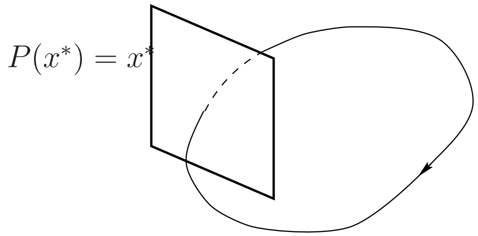







# Existence of Periodic Orbits

A point $x^*$ such that $P(x^*) = x^*$ corresponds to a periodic orbit.




$x^*$ is called a **fixed point** of $P$.





**1 minute exercise:** *What does a fixed point of $P^k$ corresponds to?*

## Stable Periodic Orbit

The linearization of $P$ around $x^*$ gives a matrix $W$ such that

$$P(x) \approx Wx$$

if $x$ is close to $x^*$.

- $\lambda_j(W) = 1$ for some $j$
- If $|\lambda_i(W)| < 1$ for all $i \neq j$, then the corresponding periodic orbit is **asymptotically stable**
- If $|\lambda_i(W)| > 1$ for some $i$, then the periodic orbit is **unstable**.





## Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= 1\end{aligned}$$

Choose $\Sigma = \{(r,\theta) : r > 0, \theta = 2\pi k\}$.

The solution is

$$\phi_t(r_0,\theta_0) = \left([1+(r_0^{-2}-1)e^{-2t}]^{-1/2}, t+\theta_0\right)$$

First return time from any point $(r_0,\theta_0) \in \Sigma$ is

$$\tau(r_0,\theta_0) = 2\pi.$$





The Poincaré map is

$$P(r_0,\theta_0) = \begin{pmatrix}[1+(r_0^{-2}-1)e^{-2\cdot 2\pi}]^{-1/2}\\ \theta_0+2\pi\end{pmatrix}$$

$(r_0,\theta_0) = (1, 2\pi k)$ is a fixed point.

The periodic solution that corresponds to $(r(t),\theta(t)) = (1,t)$ is asymptotically stable because

$$W = \frac{dP}{d(r_0,\theta_0)}(1,2\pi k) = \begin{pmatrix} e^{-4\pi} & 0\\ 0 & 1\end{pmatrix}$$

$\Rightarrow$ Stable periodic orbit (as we already knew for this example) !