2E1262 Nonlinear Control

Lecture 2

- Wrap-up of Lecture 1: Nonlinear systems and phenomena
- Modeling and simulation in Simulink
- Introduction to phase-plane analysis

Wrap-up of Lecture 1

- **Linear Systems:**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,; \quad x \in \mathbf{R}^n,\ u \in \mathbf{R}^m \\ y(t) &= Cx(t) + Du(t)\,; \quad y \in \mathbf{R}^p \end{aligned}$$

  - Closed-form solution
  - Superposition
  - Stability = global stability (unique equilibrium)
  - $u(t) = sin(\omega t)$ yields $y(t) = A_1 sin(\omega t + \phi)$

Wrap-up of Lecture 1

- **Nonlinear Systems:**

$$\begin{aligned} \dot{x} &= f(t, x, u)\,; \quad x \in \mathbf{R}^n,\ u \in \mathbf{R}^m \\ y(t) &= h(t, x, u)\,; \quad y \in \mathbf{R}^p \end{aligned}$$

  - No closed-form solution in the general case
  - Multiple equilibria, i.e., $f(t, x^*, u^*) = 0$ can have several solutions, hence local stability does not imply global stability
  - Periodic solutions $x(t) = x(t+T),\ T > 0, \dot{x} \neq 0$
  - Also aperiodic and chaotic solutions
  - Harmonic distortion: single frequency in yields multiple frequencies out

Exsistence and Uniqueness

Definition: A solution to

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \qquad (1)$$

over an interval $[0, T]$ is a $\mathbf{C}^1$ function $x : [0, T] \to \mathbb{R}^n$ such that (1) is fulfilled.

- When does there exists a solution?
- When is the solution unique?

Example: $\dot{x} = Ax$, $x(0) = x_0$, gives $x(t) = \exp(At)x_0$

Existence Problems

Example: The differential equation $\dot{x} = x^2$, $x(0) = x_0$

has solution $\quad x(t) = \dfrac{x_0}{1 - x_0 t}, \qquad 0 \le t < \dfrac{1}{x_0}$

Solution not defined for $\quad t_f = \dfrac{1}{x_0}$

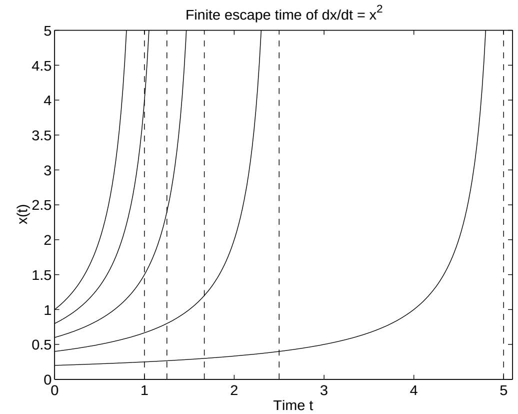
Solution interval depends on initial condition!

Recall the trick: $\dot{x} = x^2 \;\Rightarrow\; \dfrac{dx}{x^2} = dt$

Integrate $\Rightarrow \dfrac{-1}{x(t)} - \dfrac{-1}{x(0)} = t \;\Rightarrow x(t) \;=\; \dfrac{x_0}{1 - x_0 t}$

Finite Escape Time

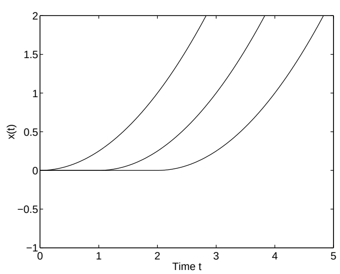
Simulation for various initial conditions x_0



Uniqueness Problems

Example: $\dot{x} = \sqrt{x}$, $x(0) = 0$, has many solutions:

$$x(t) = \begin{cases} (t-C)^2/4 & t > C \\ 0 & t \le C \end{cases}$$



Physical Interpretation

Consider the reverse example, i.e., the water tank lab process with

$$\dot{x} = -\sqrt{x}, \qquad x(T) = 0$$

where x is the water level. It is then impossible to know at what time $t < T$ the level was $x(t) = x_0 > 0$.

Hint: Reverse time $s = T - t \Rightarrow ds = -dt$ and thus

$$\frac{dx}{ds} = -\frac{dx}{dt}$$

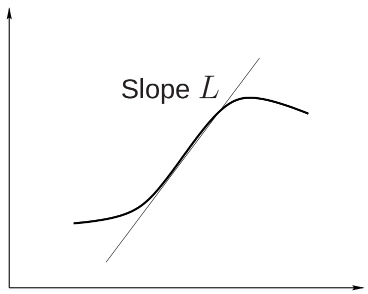
Lipschitz Continuity

Definition: $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous if there exists $L, r > 0$ such that for all
$x, y \in B_r(x_0) = \{z \in \mathbb{R}^n : \ \|z - x_0\| < r\}$,

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Euclidean norm is given by

$$\|x\|^2 = x_1^2 + \cdots + x_n^2$$



Local Existence and Uniqueness

Theorem:
If f is Lipschitz continuous, then there exists $\delta > 0$ such that

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0$$

has a unique solution in $B_r(x_0)$ over $[0, \delta]$.

Proof: See Khalil, Appendix C.1. Based on the contraction mapping theorem

Remarks

- $\delta = \delta(r, L)$
- f being C^0 is not sufficient (cf., tank example)
- f being C^1 implies Lipschitz continuity
  ($L = \max_{x \in B_r(x_0)} f'(x)$)

Today's Goal

You should be able to

- Model and simulate in Simulink
- Linearize using Simulink
- Do phase-plane analysis using pplane (or other tool)

Analysis Through Simulation

Simulation tools:

ODE's $\dot{x} = f(t, x, u)$

- ACSL, Simnon, Simulink

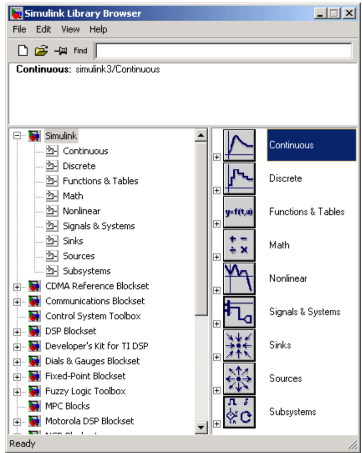
DAE's $F(t, \dot{x}, x, u) = 0$

- Omsim, Dymola, Modelica
  `http://www.modelica.org`

Special purpose simulation tools

- Spice, EMTP, ADAMS, gPROMS

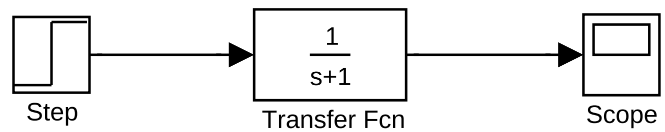
Simulink



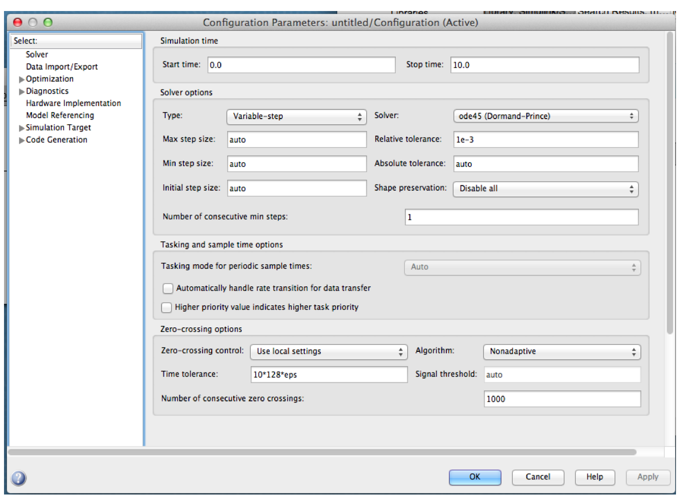

```
> matlab
>> simulink
```


An Example in Simulink

```
File -> New -> Model
Double click on Continous
Transfer Fcn
Step (in Sources)
Scope (in Sinks)
Connect (mouse-left)
Simulation -> Parameters
```



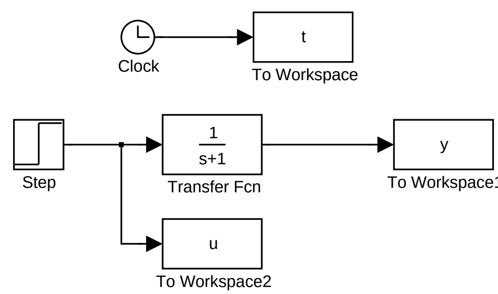
Choose Simulation Parameters



Don't forget "Apply"

Save Results to Workspace

stepmodel.mdl



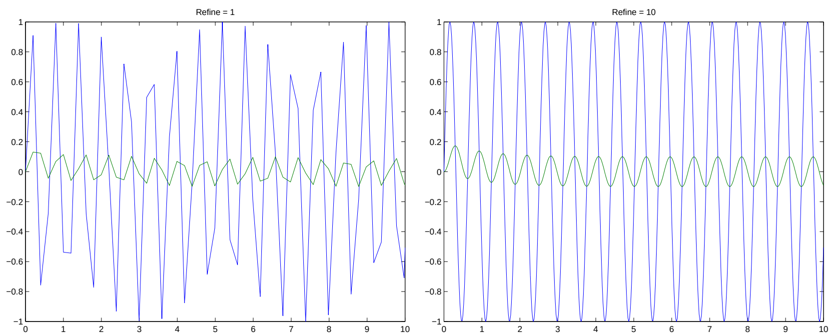
Check "Save format" of output blocks ("Array" instead of "Structure")

```
>> plot(t,y)
```

How To Get Better Accuracy

Modify Refine, Absolute and Relative Tolerances, Integration method

Refine adds interpolation points:



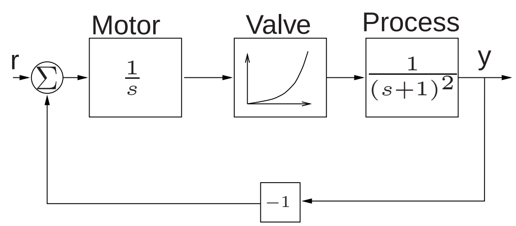
Use Scripts to Document Simulations

If the block-diagram is saved to `stepmodel.mdl`,
the following Script-file `simstepmodel.m` simulates the system:

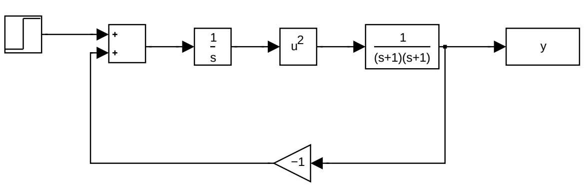
```
open_system('stepmodel')
set_param('stepmodel','RelTol','1e-3')
set_param('stepmodel','AbsTol','1e-6')
set_param('stepmodel','Refine','1')
tic
sim('stepmodel',6)
toc
subplot(2,1,1),plot(t,y),title('y')
subplot(2,1,2),plot(t,u),title('u')
```


Nonlinear Control System

Example: Control system with valve characteristic $f(u) = u^2$



Simulink block diagram:

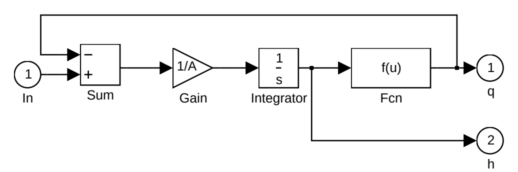


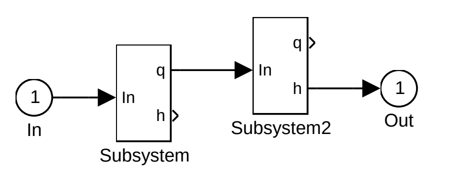
Example: Two-Tank System

The system consists of two identical tank models:

$$\dot{h} = (u - q)/A$$

$$q = a\sqrt{2g}\sqrt{h}$$





Linearization in Simulink

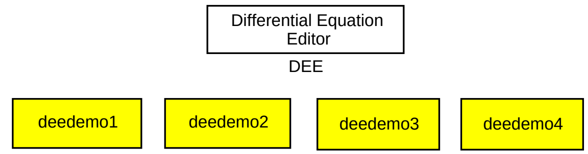
Linearize about equilibrium (x_0, u_0, y_0):

```
>> A=2.7e-3;a=7e-6,g=9.8;
>> [x0,u0,y0]=trim('twotank',[0.1 0.1]',[],0.1)
x0 =
    0.1000
    0.1000
u0 =
   9.7995e-006
y0 =
    0.1000
>> [aa,bb,cc,dd]=linmod('twotank',x0,u0);
>> sys=ss(aa,bb,cc,dd);
>> bode(sys)
```


Differential Equation Editor

`dee` is a Simulink-based differential equation editor

```
>> dee
```



Run the demonstrations

Phase-Plane Analysis

- Download ICTools from
  `http://www.control.lth.se/~ictools`
- Down load DFIELD and PPLANE from
  `http://math.rice.edu/~dfield`
  This was the preferred tool last year!

Homework 1

- Use your favorite phase-plane analysis tool
- Follow instructions in Exercise Compendium on how to write the report
- See the course homepage for a report example
- The report should be short and include only necessary plots. Write in English.