EL2620 Nonlinear Control

Automatic Control Lab, KTH

- **Disposition**

  7.5 credits, *lp 2*

  28h lectures, 28h exercises, 3 home-works

- **Instructors**

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Course Goal

To provide participants with a solid theoretical foundation of nonlinear control systems combined with a good engineering understanding

You should after the course be able to

- understand common nonlinear control phenomena
- apply the most powerful nonlinear analysis methods
- use some practical nonlinear control design methods

EL2620 Nonlinear Control

Lecture 1

- Practical information
- Course outline
- Linear vs Nonlinear Systems
- Nonlinear differential equations

Today's Goal

You should be able to

- Describe distinctive phenomena in nonlinear dynamic systems
- Mathematically describe common nonlinearities in control systems
- Transform differential equations to first-order form
- Derive equilibrium points

Course Information

- All info and handouts are available at

  `http://www.ee.kth.se/control/courses/EL2620`

- Homeworks are compulsory and have to be handed in on time
- Everyone will receive the homework of another student for review (compulsory).

Material

- **Textbook:** Khalil, *Nonlinear Systems*, Prentice Hall, 3rd ed., 2002. Optional but highly recommended.
- **Lecture notes:** Copies of transparencies (from previous year)
- **Exercises:** Class room and home exercises
- **Homeworks:** 3 computer exercises to hand in (and review)
- **Software:** Matlab

Alternative textbooks (decreasing mathematical brilliance):
Sastry, *Nonlinear Systems: Analysis, Stability and Control*; Vidyasagar, *Nonlinear Systems Analysis*; Slotine & Li, *Applied Nonlinear Control*; Glad & Ljung, *Reglerteori, flervariabla och olinjära metoder*.
Only references to Khalil will be given.

Two course compendia sold by STEX.

Course Outline

- **Introduction:** nonlinear models and phenomena, computer simulation (L1-L2)
- **Feedback analysis:** linearization, stability theory, describing functions (L3-L6)
- **Control design:** compensation, high-gain design, Lyapunov methods (L7-L10)
- **Alternatives:** gain scheduling, optimal control, neural networks, fuzzy control (L11-L13)
- **Summary** (L14)

Linear Systems

Definition: Let M be a signal space. The system $S : M \to M$ is linear if for all $u, v \in M$ and $\alpha \in \mathbb{R}$

$$S(\alpha u) = \alpha S(u) \qquad \text{scaling}$$
$$S(u + v) = S(u) + S(v) \qquad \text{superposition}$$

Example: Linear time-invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0$$
$$y(t) = g(t) \star u(t) = \int_0^t g(\tau)u(t - \tau)d\tau$$
$$Y(s) = G(s)U(s)$$

Notice the importance to have zero initial conditions

Linear Systems Have Nice Properties

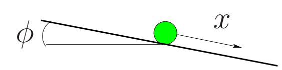
Local stability=global stability Stability if all eigenvalues of A (or poles of $G(s)$) are in the left half-plane

Superposition Enough to know a step (or impulse) response

Frequency analysis possible Sinusoidal inputs give sinusoidal outputs: $Y(i\omega) = G(i\omega)U(i\omega)$

Linear Models may be too Crude Approximations

Example: Positioning of a ball on a beam



Nonlinear model: $m\ddot{x}(t) = mg\sin\phi(t)$, Linear model: $\ddot{x}(t) = g\phi(t)$

Can the ball move 0.1 meter in 0.1 seconds from steady state?

Linear model (step response with $\phi = \phi_0$) gives

$$x(t) \approx 10\frac{t^2}{2}\phi_0 \approx 0.05\phi_0$$

so that

$$\phi_0 \approx \frac{0.1}{0.05} = 2 \text{ rad} = 114^\circ$$

Unrealistic answer. Clearly outside linear region!

Linear model valid only if $\sin\phi \approx \phi$

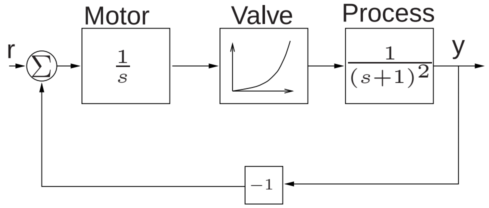
Must consider nonlinear model. Possibly also include other nonlinearities such as centripetal force, saturation, friction etc.

Linear Models are not Rich Enough

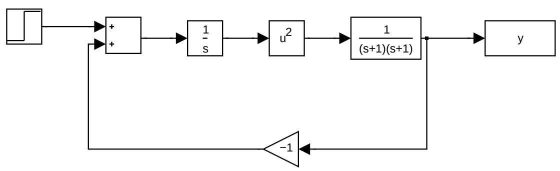
Linear models can not describe many phenomena seen in nonlinear systems

Stability Can Depend on Reference Signal

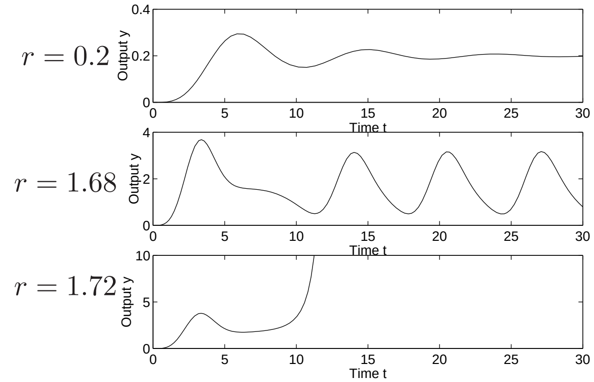
Example: Control system with valve characteristic $f(u) = u^2$



Simulink block diagram:



Step Responses



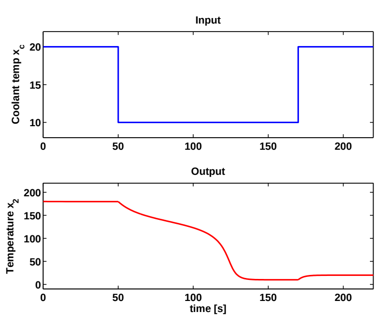
Stability depends on amplitude of the reference signal!

(The linearized gain of the valve increases with increasing amplitude)

Multiple Equilibria

Example: chemical reactor

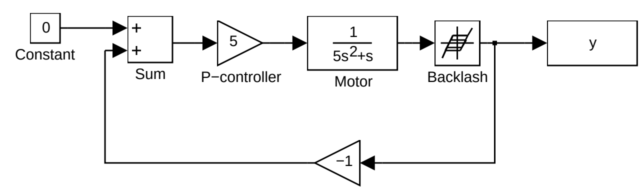
$$\begin{aligned}\dot{x}_1 &= -x_1 exp\left(-\frac{1}{x_2}\right) + f(1 - x_1)\\ \dot{x}_2 &= x_1 exp\left(-\frac{1}{x_2}\right) - \epsilon f(x_2 - x_c)\\ f &= 0.7, \epsilon = 0.4\end{aligned}$$



Existence of multiple stable equilibria for the same input gives hysteresis effect

Stable Periodic Solutions

Example: Position control of motor with back-lash

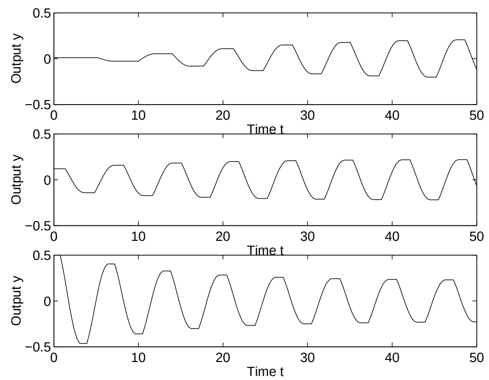


Motor: $G(s) = \frac{1}{s(1+5s)}$

Controller: $K = 5$

Back-lash induces an oscillation

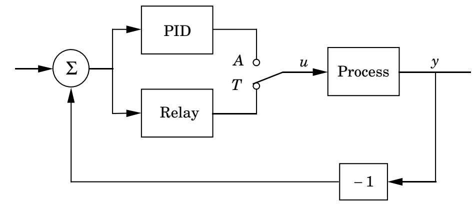
Period and amplitude independent of initial conditions:

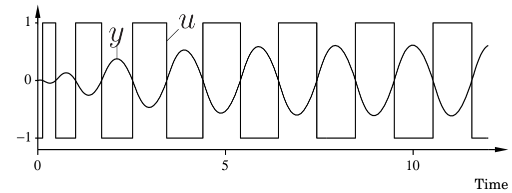


How predict and avoid oscillations?

Automatic Tuning of PID Controllers

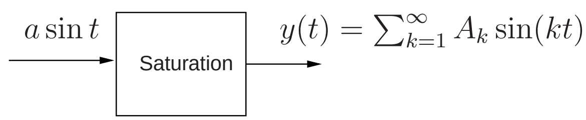
Relay induces a desired oscillation whose frequency and amplitude are used to choose PID parameters

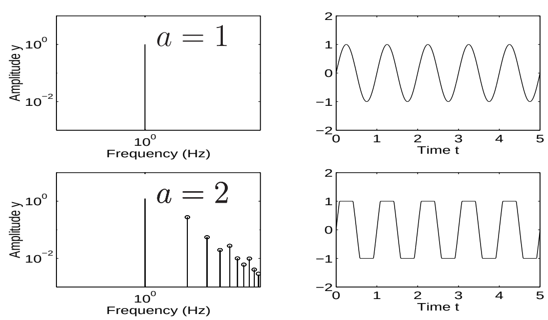




Harmonic Distortion

Example: Sinusoidal response of saturation





Example: Electrical power distribution

Nonlinearities such as rectifiers, switched electronics, and transformers give rise to harmonic distortion

$$\text{Total Harmonic Distortion} = \frac{\sum_{k=2}^{\infty} \text{Energy in tone } k}{\text{Energy in tone 1}}$$

Example: Electrical amplifiers

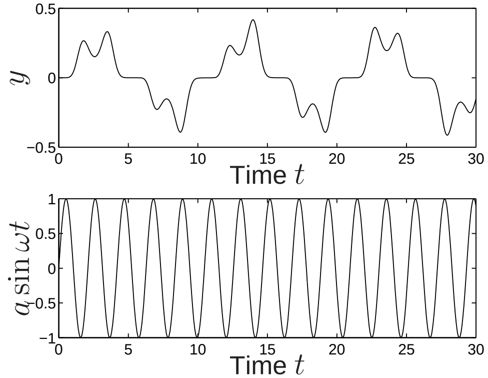
Effective amplifiers work in nonlinear region

Introduces spectrum leakage, which is a problem in cellular systems

Trade-off between effectivity and linearity

Subharmonics

Example: Duffing's equation $\ddot{y} + \dot{y} + y - y^3 = a\sin(\omega t)$



Nonlinear Differential Equations

Definition: A solution to

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \qquad (1)$$

over an interval $[0, T]$ is a $\mathbf{C}^1$ function $x : [0, T] \to \mathbb{R}^n$ such that (1) is fulfilled.

- When does there exists a solution?
- When is the solution unique?

Example: $\dot{x} = Ax$, $x(0) = x_0$, gives $x(t) = \exp(At)x_0$

Existence Problems

Example: The differential equation $\dot{x} = x^2$, $x(0) = x_0$

has solution $\quad x(t) = \dfrac{x_0}{1 - x_0 t}, \qquad 0 \le t < \dfrac{1}{x_0}$

Solution not defined for $\quad t_f = \dfrac{1}{x_0}$

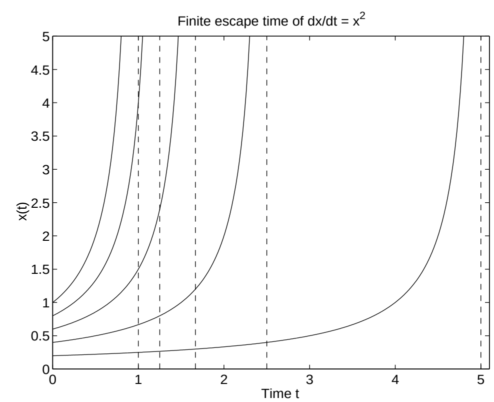
Solution interval depends on initial condition!

Recall the trick: $\dot{x} = x^2 \;\Rightarrow\; \dfrac{dx}{x^2} = dt$

Integrate $\Rightarrow \dfrac{-1}{x(t)} - \dfrac{-1}{x(0)} = t \;\Rightarrow x(t) \;=\; \dfrac{x_0}{1 - x_0 t}$

Finite Escape Time

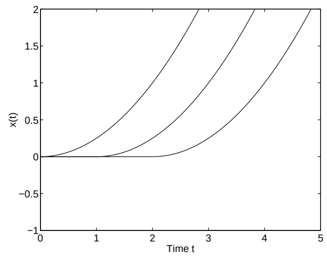
Simulation for various initial conditions x_0



Uniqueness Problems

Example: $\dot{x} = \sqrt{x}$, $x(0) = 0$, has many solutions:

$$x(t) = \begin{cases} (t-C)^2/4 & t > C \\ 0 & t \leq C \end{cases}$$



Physical Interpretation

Consider the reverse example, i.e., the water tank lab process with

$$\dot{x} = -\sqrt{x}, \qquad x(T) = 0$$

where x is the water level. It is then impossible to know at what time $t < T$ the level was $x(t) = x_0 > 0$.

Hint: Reverse time $s = T - t \Rightarrow ds = -dt$ and thus

$$\frac{dx}{ds} = -\frac{dx}{dt}$$

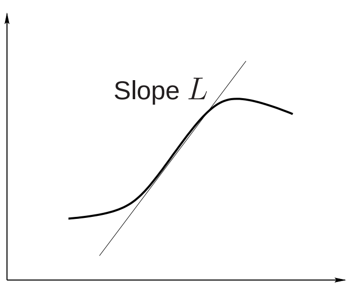
Lipschitz Continuity

Definition: $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous if there exists $L, r > 0$ such that for all
$x, y \in B_r(x_0) = \{z \in \mathbb{R}^n : \ \|z - x_0\| < r\}$,

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Euclidean norm is given by

$$\|x\|^2 = x_1^2 + \cdots + x_n^2$$



Local Existence and Uniqueness

Theorem:
If f is Lipschitz continuous, then there exists $\delta > 0$ such that

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0$$

has a unique solution in $B_r(x_0)$ over $[0, \delta]$. **Proof:** See Khalil, Appendix C.1. Based on the contraction mapping theorem

Remarks

- $\delta = \delta(r, L)$
- f being C^0 is not sufficient (cf., tank example)
- f being C^1 implies Lipschitz continuity
  ($L = \max_{x \in B_r(x_0)} f'(x)$)

State-Space Models

State x, input u, output y

General: $f(x, u, y, \dot{x}, \dot{u}, \dot{y}, \ldots) = 0$

Explicit: $\dot{x} = f(x, u), \quad y = h(x)$

Affine in u: $\dot{x} = f(x) + g(x)u, \quad y = h(x)$

Linear: $\dot{x} = Ax + Bu, \quad y = Cx$

Transformation to Autonomous System

A nonautonomous system

$$\dot{x} = f(x, t)$$

is always possible to transform to an autonomous system by introducing $x_{n+1} = t$:

$$\dot{x} = f(x, x_{n+1})$$
$$\dot{x}_{n+1} = 1$$

Transformation to First-Order System

Given a differential equation in y with highest derivative $\frac{d^n y}{dt^n}$, express the equation in $x = \begin{pmatrix} y & \frac{dy}{dt} & \ldots & \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}^T$ **Example**:

Pendulum

$$MR^2\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

$x = \begin{pmatrix} \theta & \dot{\theta} \end{pmatrix}^T$ gives

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{MR^2}x_2 - \frac{g}{R}\sin x_1$$

Equilibria

Definition: A point (x^*, u^*, y^*) is an equilibrium, if a solution starting in (x^*, u^*, y^*) stays there forever.

Corresponds to putting all derivatives to zero:

General: $f(x^*, u^*, y^*, 0, 0, \ldots) = 0$

Explicit: $0 = f(x^*, u^*), \quad y^* = h(x^*)$

Affine in u: $0 = f(x^*) + g(x^*)u^*, \quad y^* = h(x^*)$

Linear: $0 = Ax^* + Bu^*, \quad y^* = Cx^*$

Often the equilibrium is defined only through the state x^*

Multiple Equilibria

Example: Pendulum

$$MR^2\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

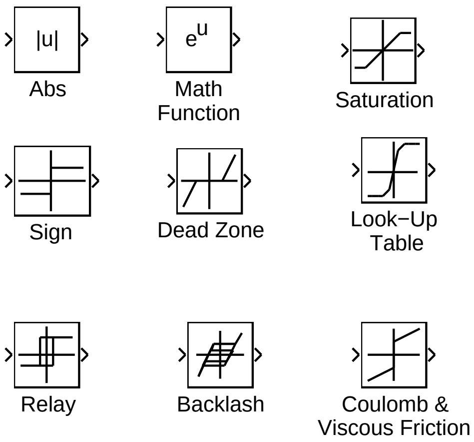
$\ddot{\theta} = \dot{\theta} = 0$ gives $\sin\theta = 0$ and thus $\theta^* = k\pi$

Alternatively in first-order form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{MR^2}x_2 - \frac{g}{R}\sin x_1$$

$\dot{x}_1 = \dot{x}_2 = 0$ gives $x_2^* = 0$ and $\sin(x_1^*) = 0$

Some Common Nonlinearities in Control Systems



When do we need Nonlinear Analysis & Design?

- When the system is strongly nonlinear
- When the range of operation is large
- When distinctive nonlinear phenomena are relevant
- When we want to push performance to the limit

Next Lecture

- Simulation in Matlab
- Linearization
- Phase plane analysis