VEKTORANALYS

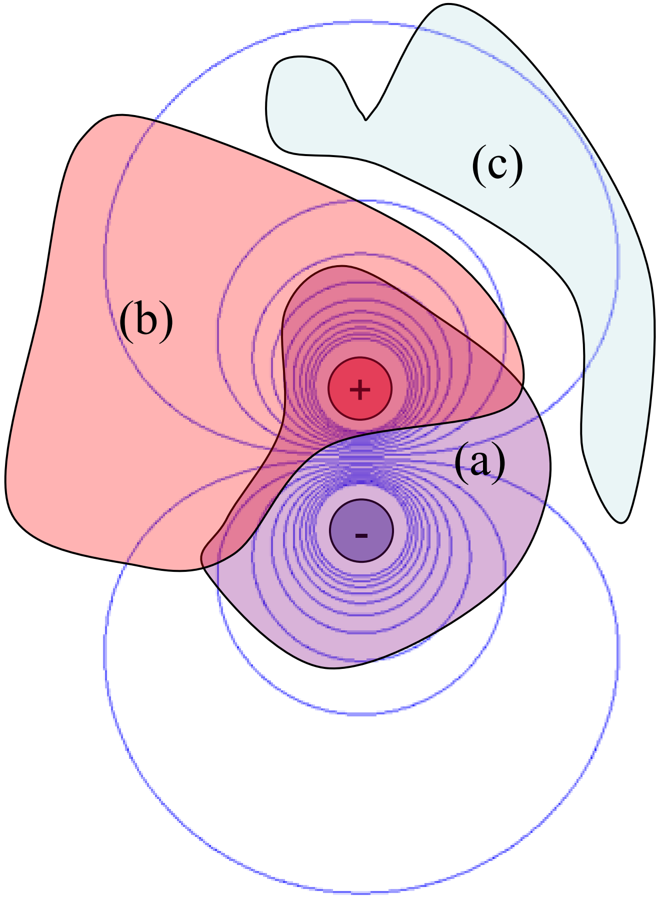
Kursvecka 6

övningar

PROBLEM 1

A dipole is formed by two point sources with charge +c and -c
Calculate the flux of the dipole field on a closed surface S that

(a) encloses both poles
(b) encloses only the plus pole
(c) does not enclose any pole



SOLUTION

Vector field from a point source located in $\overline{r}_0$*:*

$$\overline{A}(\overline{r}) = c\frac{\overline{r}-\overline{r}_0}{\left|\overline{r}-\overline{r}_0\right|^3}$$

Vector field from two point sources located in $\overline{r}_1$ *and* $\overline{r}_2$

$$\overline{A}(\overline{r}) = c\frac{\overline{r}-\overline{r}_1}{\left|\overline{r}-\overline{r}_1\right|^3} - c\frac{\overline{r}-\overline{r}_2}{\left|\overline{r}-\overline{r}_2\right|^3}$$

Flux from a point source:

$$= \begin{cases} 0 & \textit{If the origin is outside V} \\ 4\pi c & \textit{If the origin is inside V} \end{cases}$$

$$\oiint_S \left(c\frac{\overline{r}-\overline{r}_1}{\left|\overline{r}-\overline{r}_1\right|^3} - c\frac{\overline{r}-\overline{r}_2}{\left|\overline{r}-\overline{r}_2\right|^3} \right)\cdot d\overline{S} = \oiint_S c\frac{\overline{r}-\overline{r}_1}{\left|\overline{r}-\overline{r}_1\right|^3}\cdot d\overline{S} - \oiint_s c\frac{\overline{r}-\overline{r}_2}{\left|\overline{r}-\overline{r}_2\right|^3}\cdot d\overline{S}$$

(a) $\oiint_S (\ )\cdot d\overline{S} = \quad 4\pi c \quad -4\pi c \quad =0$

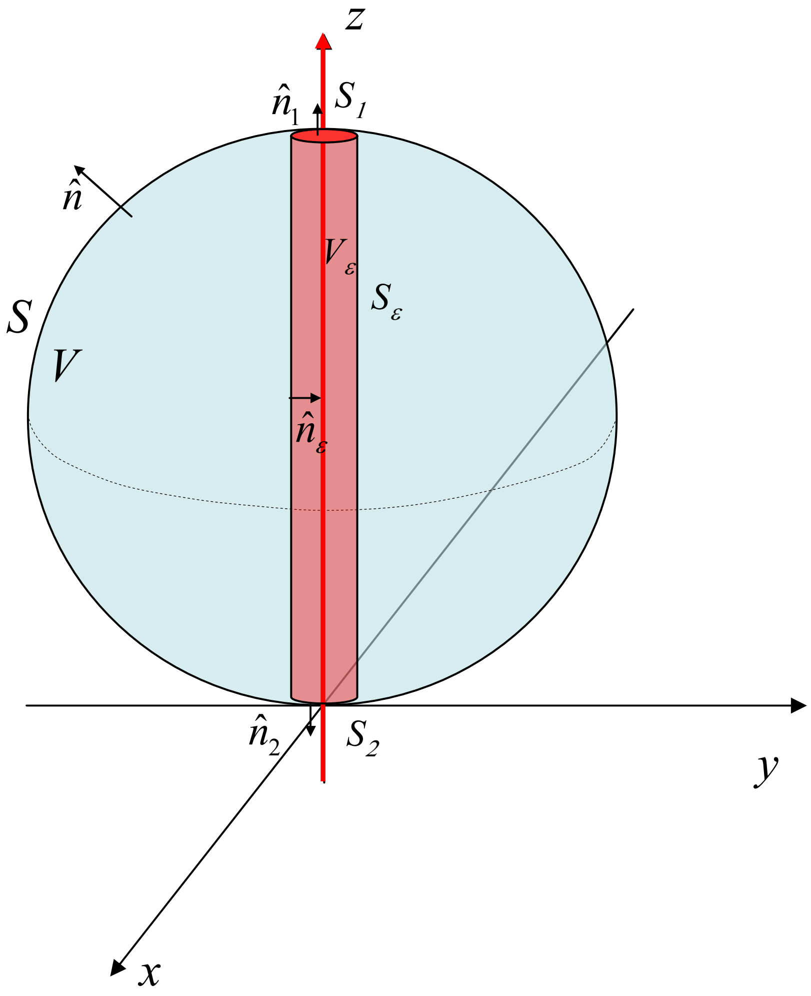
(b) $\oiint_S (\ )\cdot d\overline{S} = \quad 4\pi c \quad 0 \quad = 4\pi c$

(c) $\oiint_S (\ )\cdot d\overline{S} = \quad 0 \quad 0 \quad = 0$

PROBLEM 2

Use the Gauss theorem to calculate the flux of the vector field: $\bar{A}(\rho,\varphi,z) = z\dfrac{\rho^2-1}{\rho}\hat{e}_\rho$

on the surface S: $x^2+y^2+(z-2)^2 \le 4$



SOLUTION

The field is singular at ρ=0 (the z-axis)
The Gauss theorem cannot be applied on S!!!

We divide V into 2 volumes:

$V=V_0+V_\varepsilon$

Thin cylinder with radius ε along the z-axis

The boundary surface to V_0 is: $S+S_\varepsilon-S_1-S_2$

Does not contain the z-axis
$\Rightarrow$ here we can apply Gauss!

$$\iint_S \bar{A}(\bar{r})\cdot d\bar{S} = \iint_{\substack{S+S_\varepsilon-S_\varepsilon+S_1\\ -S_1+S_2-S_2}} \bar{A}(\bar{r})\cdot d\bar{S} =$$

$$= \iint_{S+S_\varepsilon-S_1-S_2} \bar{A}(\bar{r})\cdot d\bar{S} + \iint_{-S_\varepsilon+S_1+S_2} \bar{A}(\bar{r})\cdot d\bar{S} = \iiint_{V_0} div\bar{A}dV + \iint_{-S_\varepsilon+S_1+S_2} \bar{A}(\bar{r})\cdot d\bar{S}$$

$$= \iiint\limits_{V_0} div\bar{A}dV + \iint\limits_{-S_\varepsilon + S_1 + S_2} \bar{A}(\bar{r})\cdot d\bar{S} = \underset{I}{\iiint\limits_{V_0} div\bar{A}dV} - \underset{II}{\iint\limits_{S_\varepsilon} \bar{A}(\bar{r})\cdot d\bar{S}} + \underset{III}{\iint\limits_{S_1} \bar{A}(\bar{r})\cdot d\bar{S}} + \underset{IV}{\iint\limits_{S_2} \bar{A}(\bar{r})\cdot d\bar{S}}$$

Integrals *III* and *IV* are zero:

$$\iint\limits_{S_1} \bar{A}(\bar{r})\cdot d\bar{S} = \iint\limits_{S_1} z\frac{\rho^2-1}{\rho}\underbrace{\hat{e}_\rho\cdot\hat{e}_z}_{=0}dS = \boxed{0}$$

$$\lim_{\varepsilon\to 0} d\bar{S} = \hat{e}_z dS$$

$$\iint\limits_{S_2} \bar{A}(\bar{r})\cdot d\bar{S} = -\iint\limits_{S_2} z\frac{\rho^2-1}{\rho}\underbrace{\hat{e}_\rho\cdot\hat{e}_z}_{=0}dS = \boxed{0}$$

$$\lim_{\varepsilon\to 0} d\bar{S} = -\hat{e}_z dS$$

Integrals *II* is:

$$d\bar{S} = -\hat{e}_\rho dS \qquad dS = \varepsilon d\varphi dz \qquad On\ S_\varepsilon\ \rho=\varepsilon$$

$$-\iint\limits_{S_\varepsilon} \bar{A}(\bar{r})\cdot d\bar{S} = -\iint\limits_{S_\varepsilon} z\frac{\rho^2-1}{\rho}\hat{e}_\rho\cdot\left(-\hat{e}_\rho\right)dS = \iint\limits_{S_\varepsilon} z\frac{\rho^2-1}{\rho}dS = \iint\limits_{S_\varepsilon} z\frac{\varepsilon^2-1}{\varepsilon}\varepsilon d\varphi dz =$$

$$= \int\limits_0^{2\pi}\int\limits_{2-\sqrt{4-\varepsilon^2}}^{2-\sqrt{4-\varepsilon^2}} z\left(\varepsilon^2-1\right)d\varphi dz = 2\pi\left(\varepsilon^2-1\right)\int\limits_{2-\sqrt{4-\varepsilon^2}}^{2+\sqrt{4-\varepsilon^2}} zdz = 8\pi\left(\varepsilon^2-1\right)\sqrt{4-\varepsilon^2}$$

But our cylinder is very *“thin”*, which means that we are interested in $\lim\limits_{\varepsilon\to 0}$

$$\lim_{\varepsilon\to 0} 8\pi\left(\varepsilon^2-1\right)\sqrt{4-\varepsilon^2} = \boxed{-16\pi}$$

Integrals I is:

$$\iiint_{V_0} div\bar{A}dV = \iiint_{V_0} div\left(z\frac{\rho^2-1}{\rho}\hat{e}_\rho\right)dV = \iiint_{V_0}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho z\frac{\rho^2-1}{\rho}\right)dV = \iiint_{V_0} 2zdV$$

The volume is a sphere with centre in the point (0,0,2)
Therefore, to use spherical coord. we must do a coordinate transformation: z’=z-2

$$\iiint_{V_0}2zdV = \iiint_{V_0}2(z'+2)dV = \iiint_{V_0}4dV + \iiint_{V_0}2z'dV = 4\frac{4\pi 2^3}{3} + \iiint_{V_0}2r\cos\theta r^2\sin\theta drd\theta d\varphi$$

$$=\frac{128\pi}{3}+2\left[\frac{r^4}{4}\right]_0^2 2\pi\underbrace{\left[-\frac{\cos^2\theta}{2}\right]_0^\pi}_{0} = \boxed{\frac{128\pi}{3}}$$

$$I+II+III+IV = \frac{128\pi}{3}-16\pi+0+0=-\frac{80\pi}{3}$$

LAPLACE OPERATOR IN CURVILINEAR COORDINATES

The Laplace equation is $\nabla^2\phi = 0$

The Laplace operator ∇^2 is the divergence of the gradient: $\boxed{\nabla^2\phi \equiv \nabla\cdot\nabla\phi}$

CARTESIAN COORDINATES

- GRADIENT $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$
- DIVERGENCE $\nabla\cdot\bar{A} = \frac{\partial A_z}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

⇨

CARTESIAN

$$\boxed{\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)}$$

CYLINDRICAL COORDINATES

- GRADIENT $\nabla\phi = \left(\frac{\partial\phi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}, \frac{\partial\phi}{\partial z}\right)$
- DIVERGENCE $\nabla\cdot\bar{A} = \left(\frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}\right)$

⇨

CYLINDRICAL

$$\boxed{\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}}$$

SPHERICAL COORDINATES

- GRADIENT $\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$
- DIVERGENCE $\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)$

⇨

SPHERICAL

$$\boxed{\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}}$$

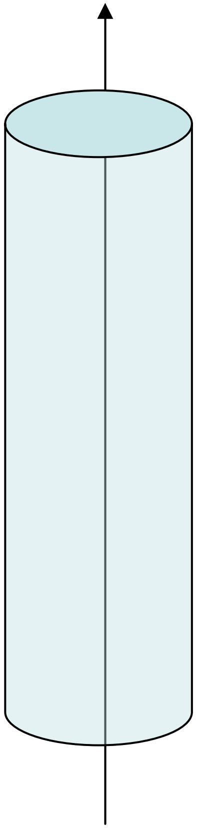
PROBLEM 3

A cylinder with radius R has a charge density ρ_c.
Calculate the potential and the electric field:

(a) Inside the cylinder
(b) Outside the cylinder

Assume that the cylinder is infinitely long and that the potential on the surface is V_o.

SOLUTION



Due to the symmetry of the problem,
the solution will depend on the radius only: $\phi=\phi(\rho)$

Inside the cylinder

$$\nabla^2\phi = -\frac{\rho_c}{\varepsilon_0}$$

cylindrical coord.

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \underbrace{\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2}}_{=0} + \underbrace{\frac{\partial^2\phi}{\partial z^2}}_{=0} = -\frac{\rho_c}{\varepsilon_0}$$

← *Because the solution depends only on ρ*

The equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) = -\frac{\rho_c}{\varepsilon_0}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=-\frac{\rho_c}{\varepsilon_0}\overset{\text{Multiplying by }\rho}{\Rightarrow}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=-\frac{\rho_c}{\varepsilon_0}\rho\overset{\text{Integrating in }\rho}{\Rightarrow}\rho\frac{\partial\phi}{\partial\rho}=-\frac{\rho_c}{2\varepsilon_0}\rho^2+a\overset{\text{Dividing by }\rho}{\Rightarrow}\frac{\partial\phi}{\partial\rho}=-\frac{\rho_c}{2\varepsilon_0}\rho+\frac{a}{\rho}\Rightarrow$$

$$\left[\bar{E}=-\nabla\phi=-\left(\frac{d\phi(\rho)}{d\rho},\frac{1}{\rho}\underbrace{\frac{d\phi(\rho)}{d\varphi}}_{=0},\underbrace{\frac{d\phi(\rho)}{dz}}_{=0}\right)\Rightarrow E_r=-\frac{d\phi(\rho)}{d\rho}=\boxed{+\frac{\rho_c}{2\varepsilon_0}\rho}-\frac{a}{\rho}\right]$$

Because the solution depends only on ρ

Divergent at $\rho=0$ NOT physical! $\Rightarrow$ $a=0$

$$\frac{\partial\phi}{\partial\rho}=-\frac{\rho_c}{2\varepsilon_0}\rho\underset{\text{Integrating in }\rho}{\Rightarrow}\boxed{\phi(\rho)=-\frac{\rho_c}{4\varepsilon_0}\rho^2+b}$$

Outside the cylinder

$$\nabla^2\phi=0 \leftarrow \text{There is no charge}$$

And the equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=0\underset{\text{Multiplying by }\rho}{\Rightarrow}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)=0\underset{\text{Integrating in }\rho}{\Rightarrow}\rho\frac{\partial\phi}{\partial\rho}=c\underset{\text{Dividing by }\rho}{\Rightarrow}\frac{\partial\phi}{\partial\rho}=\frac{c}{\rho}\underset{\text{Integrating in }\rho}{\Rightarrow}\boxed{\phi(\rho)=c\ln\rho+d}$$

$$E_r=-\frac{d\phi(\rho)}{d\rho}=\boxed{-\frac{c}{\rho}}$$

$$E_r^{in}(\rho) = +\frac{\rho_c}{2\varepsilon_0}\rho \qquad \phi^{in}(\rho) = -\frac{\rho_c}{4\varepsilon_0}\rho^2 + b$$

$$E_r^{out} = -\frac{c}{\rho} \qquad \phi^{out}(\rho) = c\ln\rho + d$$

Now we must determine the three integration constants *b, c* and *d*.
We have three conditions:
(1) Continuity of the electric field at ρ =R
(2) The potential at ρ =R
(3) Continuity of the potential at ρ =R

$$(1) \qquad E_r^{in}(R) = E_r^{out}(R) \;\Rightarrow\; +\frac{\rho_c}{2\varepsilon_0}R = -\frac{c}{R} \;\Rightarrow\; \boxed{c = -\frac{\rho_c}{2\varepsilon_0}R^2} \qquad \Rightarrow \phi^{out}(\rho) = -\frac{\rho_c}{2\varepsilon_0}R^2\ln\rho + d$$

$$(2) \qquad \phi^{out}(R) = V_0 \;\Rightarrow\; -\frac{\rho_c}{2\varepsilon_0}R^2\ln R + d = V_0 \;\Rightarrow\; \boxed{d = V_0 + \frac{\rho_c}{2\varepsilon_0}R^2\ln R}$$

$$\Rightarrow \phi^{out}(\rho) = -\frac{\rho_c}{2\varepsilon_0}R^2\ln\rho + d = -\frac{\rho_c}{2\varepsilon_0}R^2\ln\rho + V_0 + \frac{\rho_c}{2\varepsilon_0}R^2\ln R = V_0 - \frac{\rho_c}{2\varepsilon_0}R^2\ln\left(\frac{\rho}{R}\right)$$

$$(3) \qquad \phi^{out}(R) = \phi^{in}(R) \;\Rightarrow\; V_0 - \frac{\rho_c}{2\varepsilon_0}R^2\underbrace{\ln\left(\frac{R}{R}\right)}_{=0} = -\frac{\rho_c}{4\varepsilon_0}R^2 + b \;\Rightarrow\; \boxed{b = V_0 + \frac{\rho_c}{4\varepsilon_0}R^2}$$

$$\phi^{in}(\rho) = V_0 + \frac{\rho_c}{4\varepsilon_0} R^2 \left(1 - \frac{\rho^2}{R^2}\right) \qquad E_r^{in}(\rho) = +\frac{\rho_c}{2\varepsilon_0}\rho$$

$$\phi^{out}(\rho) = V_0 - \frac{\rho_c}{2\varepsilon_0} R^2 \ln\left(\frac{\rho}{R}\right) \qquad E_r^{out} = \frac{\rho_c}{2\varepsilon_0}\frac{R^2}{\rho}$$