SF2822 Applied nonlinear optimization, final exam
Monday August 19 2024 8.00–13.00
Brief solutions

1. As $g_3(x^*) > 0$ we must have $g_3(x) \geq 0$. Therefore, by complementarity we must have $\lambda_3 = 0$ in first-order optimality conditions.

Since $g_1(x^*) = 0$, $g_2(x^*) = 0$, with $\nabla g_1(x^*)$ and $\nabla g_2(x^*)$ linearly independent, it follows that x^* is a regular point. Hence, the first-order necessary optimality conditions must hold. We therefore try to find λ_1 and λ_2 such that

$$\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \lambda_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \lambda_2.$$

There is a unique solution given by $\lambda_1 = 2$ and $\lambda_2 = -3$. Since $\lambda_1 > 0$ and $\lambda_2 < 0$, we must have $g_1(x) \geq 0$ and $g_2(x) \leq 0$ for the first-order necessary optimality conditions to hold.

We now investigate whether this choice gives a local minimizer. The Jacobian of the active constraints at x^* is given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

As the first two columns form an invertible matrix, we may for example obtain Z from

$$Z = \begin{pmatrix} -\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence,

$$Z^T(\nabla^2 f(x^*) - \lambda_1 \nabla^2 g_1(x^*) - \lambda_2 \nabla^2 g_2(x^*))Z = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1,$$

which is not a positive semidefinite matrix. Therefore, x^* is a regular point at which strict complementarity holds, and the second-order sufficient optimality do not hold. Therefore, x^* is not a local minimizer to (NLP). Consequently, there is no choice of "?" such that x^* is a local minimizer to (NLP).

2. (a) The quadratic programming subproblems must have nonnegative values on λ. Since this is not the case in the prinout, the prinout cannot be correct.

(b) We have

$$f(x) = e^{x_1} + \frac{1}{2}(x_1 + x_2 - 4)^2 + (x_1 - x_2)^2,$$

$$g(x) = -(x_1 - 3)^2 - x_2^2 + 9,$$

$$\nabla f(x) = \begin{pmatrix} e^{x_1} + 3x_1 - x_2 - 4 \\ -x_1 + 3x_2 - 4 \end{pmatrix}, \qquad \nabla g(x) = \begin{pmatrix} -2(x_1 - 3) \\ -2x_2 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} e^{x_1} + 3 & -1 \\ -1 & 3 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Insertion of numerical values in the expressions above gives the first QP-problem according to

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T H p + c^T p \\ \text{subject to} & Ap \geq b, \end{array}$$

with

$$H = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}, \quad c = \begin{pmatrix} -3 \\ -4 \end{pmatrix}, \quad A = \begin{pmatrix} 6 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \end{pmatrix}.$$

This is a convex quadratic program. If we guess that the constraint is inactive, we obtain

$$p = -H^{-1}c = \begin{pmatrix} \frac{13}{11} \\ \frac{19}{11} \end{pmatrix}.$$

For this p, it holds that $Ap \geq b$, and hence we have the optimal solution to the QP-problem, with $\lambda = 0$.

(c) The fact that the λ components from the prinout are negative suggests that the inequality constraint is incorrectly treated as an equality, i.e., the printout corresponds to

$$\begin{array}{ll} \text{minimize} & e^{x_1} + \frac{1}{2}(x_1 + x_2 - 4)^2 + (x_1 - x_2)^2 \\ \text{subject to} & -(x_1 - 3)^2 - x_2^2 + 9 = 0. \end{array}$$

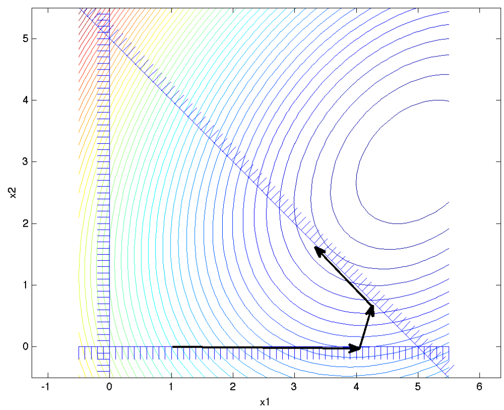
It can be seen that the norm of the gradient of the Lagrangian converges to zero. We do not have printout of the constraint, but may estimate

$$-(0.7984 - 3)^2 - 2.0378^2 - 9 \approx -2.2^2 - 2.0^2 + 9 = 0.16.$$

The constraint value is zero to numerical precision if more digits are added in the calculation. Not required as you do not have calculator. Therefore, we conclude that the first-order optimality conditions are satisfied.

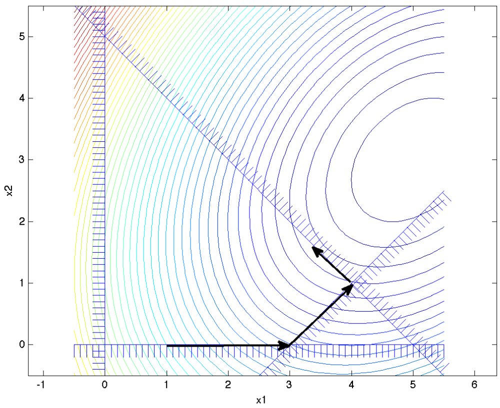
Alternatively, the same conclusion could be drawn by assuming that the inequality had been set to $-(x_1 - 3)^2 - x_2^2 + 9 \leq 0$.

3. (a) The iterations are illustrated in the figure below:



In the first iteration the search direction points at $(4\ 0)^T$, which is feasible. At this point, the multiplier of the constraint $x_2 \geq 0$ is negative, and the constraint is deleted from the active set. In the second iteration, the search direction points at $(5\ 3)^T$, but is limited by the constraint $-x_1 - x_2 \geq -5$, which is added. The search direction now points at $(7/2\ 3/2)^T$, which is feasible. The multiplier is positive, and the problem is thus solved.

(b) The iterations are illustrated in the figure below:



In the first iteration the search direction points at $(4\ 0)^T$, but the step is limited by the constraint $x_2 - x_1 \geq -3$, which is added. A zero step is taken, and the multiplier for the constraint $x_2 \geq 0$ is negative. This constraint is deleted. The new step is limited by the constraint $-x_1 - x_2 \geq -5$, which is added. A zero step is taken, and the multiplier for the constraint $x_2 - x_1 \geq -3$ is negative. This constraint is deleted, and the new step leads to the point $(7/2\ 3/2)^T$, which is feasible. The multiplier is positive, and the problem is thus solved.

4. (See the course material.)

5. (a) The relaxed problem is a non-convex quadratic programming problem. To obtain a lower bound of the original problem we do need to calculate a global minimizer of this non-convex relaxed problem, which in general is not computationally tractable.

(b) If we let (SDP') be the problem arising as the constraint $Y = xx^T$ is added to (SDP) we can replace Y with xx^T, which by (i) gives

$$(SDP') \quad \begin{array}{ll} \min & c^Tx + \frac{1}{2}x^THx \\ \text{subject to} & \begin{pmatrix} xx^T & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ & x_j^2 = x_j, \quad j = 1, \dots, n. \end{array}$$

By hint (ii) we can see that the constraint

$$\begin{pmatrix} xx^T & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is always fulfilled, hence (SDP') may be written as

$$(SDP') \quad \begin{array}{ll} \min & c^Tx + \frac{1}{2}x^THx \\ & x_j^2 = x_j, \quad j = 1, \dots, n. \end{array}$$

But $x_j^2 = x_j$ if and only if $x_j \in \{0, 1\}$. Hence, (SDP') and (P) are equivalent.