SF2822 Applied nonlinear optimization, final exam
Monday August 19 2024 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear programing problem

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \; ? \; 0, \quad i = 1, \dots, 3, \\ & x \in I\!R^3, \end{array}$$

where $f : I\!R^3 \to I\!R$ and $g : I\!R^3 \to I\!R^3$ are twice continuously differentiable and each "?" is an inequality, either "$\le$" or "$\ge$". The inequalities can be of different type for the different constraints.

Assume that we have a point x^* such that

$$f(x^*) = 3, \quad \nabla f(x^*) = \begin{pmatrix} 2 & -5 & 3 \end{pmatrix}^T, \quad \nabla^2 f(x^*) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$g_1(x^*) = 0, \quad \nabla g_1(x^*) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T, \quad \nabla^2 g_1(x^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$g_2(x^*) = 0, \quad \nabla g_2(x^*) = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T, \quad \nabla^2 g_2(x^*) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g_3(x^*) = 1, \quad \nabla g_3(x^*) = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}^T, \quad \nabla^2 g_3(x^*) = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Is it possible to replace each "?" by either a "$\le$" or a "$\ge$" so that x^* becomes a local minimizer to (NLP)? .... (10p)

2. Consider the nonlinear optimization problem (NLP) defined as

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & e^{x_1} + \frac{1}{2}(x_1 + x_2 - 4)^2 + (x_1 - x_2)^2 \\ \text{subject to} & -(x_1 - 3)^2 - x_2^2 + 9 \geq 0. \end{array}$$

You have obtained a printout from a sequential quadratic programming solver for this problem. The initial point is $x = (0\ 0)^T$ and $\lambda = 0$. Six iterations, without linesearch, have been performed. The printout, where the floating point numbers are given with four decimal places, reads:

It	x_1	x_2	λ	$\|\nabla f(x) - \nabla g(x)\lambda\|$
0	0	0	0	5.0000
1	0	1.3333	−0.7222	1.9259
2	1.0117	2.9430	−0.5596	1.1891
3	0.7084	2.1240	−0.4384	0.2028
4	0.7990	2.0422	−0.3294	0.0335
5	0.7983	2.0378	−0.3227	0.0001
6	0.7984	2.0378	−0.3227	0.0000

(a) Why cannot the above printout be correct? Give a reason that does not need any calculations in addition to the printout above. .... (2p)

(b) Formulate the first QP problem. Solve this QP problem by any method, that need not be systematic. .... (6p)

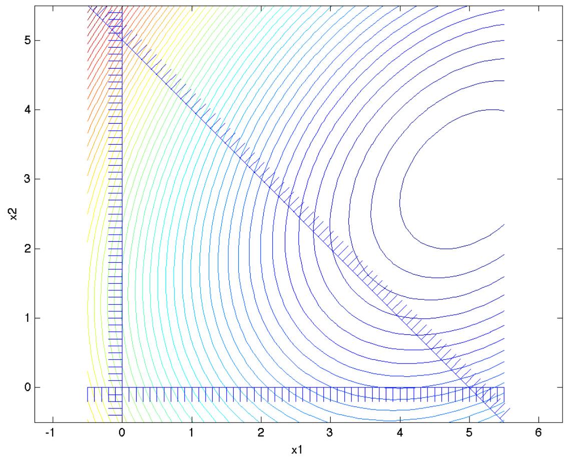
(c) All numerical values given in the table are correct for a problem related to (NLP). Make a qualified guess as to which problem this might be and motivate the answer. .... (2p)

Note: According to the convention of the book we define the Lagrangian $\mathcal{L}(x, \lambda)$ as $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ the objective function and $g(x)$ is the constraint function, where the inequality constraint is written as $g(x) \geq 0$.

3. Consider the quadratic program (QP) defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & 3x_1^2 - 2x_1x_2 + 3x_2^2 - 24x_1 - 8x_2 \\ \text{subject to} & -x_1 - x_2 \geq -5, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{array}$$

The problem may be illustrated geometrically in the figure below,



(a) Solve (QP) by an active-set method. Start at $x = (1 \ \ 0)^T$ with the constraint $x_2 \geq 0$ active. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Exercise 3a which can be found at the last sheet of the exam. Motivate each step carefully. ....(4p)

(b) Assume that the constraint $x_2 - x_1 \geq -3$ is added to (QP), so that we obtain the problem (QP') according to

$$(QP') \quad \begin{array}{ll} \text{minimize} & 3x_1^2 - 2x_1x_2 + 3x_2^2 - 24x_1 - 8x_2 \\ \text{subject to} & -x_1 - x_2 \geq -5, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_2 - x_1 \geq -3. \end{array}$$

Solve (QP') by an active-set method. Start at $x = (1 \ \ 0)^T$ with the constraint $x_2 \geq 0$ active. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Add the constraint $x_2 - x_1 \geq -3$ and illustrate your iterations in the figure corresponding to Exercise 3b which can be found at the last sheet of the exam. Motivate each step carefully. ....(6p)

4. Consider the nonlinear programming problem

$$(P) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are continuously differentiable.

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(g_i(x)).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\nabla f(x) - \nabla g(x)\lambda = 0,$$
$$g_i(x)\lambda_i - \mu = 0, \quad i = 1, \dots, m,$$

assuming that $g(x) > 0$ and $\lambda > 0$ is kept implicitly. .... (4p)

(b) Let $x(\mu)$, $\lambda(\mu)$ be a solution to the primal-dual nonlinear equations of (4a) such that $g_i(x(\mu)) > 0$, $i = 1, \dots, m$, and $\lambda(\mu) > 0$. Show that $x(\mu)$ is a global minimizer to (P_μ) if f and $-g_i$, $i = 1, \dots, m$, are convex functions on $\mathbb{R}^n$. (2p)

(c) Derive the system of linear equations that results when the primal-dual nonlinear equations of (4a) are solved by Newton's method. .... (4p)

5. Consider the optimization problem (P) defined by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & x_j \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

where H is an indefinite symmetric matrix. Problems of this type arise within combinatorial optimization, and the interest is to find a global minimizer.

One may compute lower bounds on the optimal value of (P) by considering relaxed problems.

(a) One way to relax (P) is to replace the constraints $x_j \in \{0, 1\}$, $j = 1, \dots, n$, with $0 \le x_j \le 1$, $j = 1, \dots, n$. This gives a relaxed problem without discrete variables, according to

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & 0 \le x_j \le 1, \quad j = 1, \dots, n, \end{array}$$

Explain way this relaxed problem is not very interesting in practise. .... (3p)

(b) An alternative way to create a relaxation to (P) is to introduce a symmetric matrix Y and formulate the semidefinite programming problem

$$(SDP) \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2}\,\text{trace}(HY) \\ \text{subject to} & \begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ & Y = Y^T, \\ & y_{jj} = x_j, \quad j = 1, \dots, n. \end{array}$$

Show that if the constraint $Y = xx^T$ is added to (SDP), one obtains a problem which is equivalent to (P). .... (7p)

Hint: The following two results, which may be used without proof, might be useful:

(i) If H is an $n \times n$-matrix and x is an n-vector, then $\text{trace}(Hxx^T) = x^T H x$.

(ii) If Y is a symmetric $n \times n$-matrix and x is an n-vector, then

$$\begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if and only if} \quad Y - xx^T \succeq 0.$$

Good luck!

Name: . . . . Personal number: . . . . Sheet nummer: . . . .

Figure for Exercise 3a:

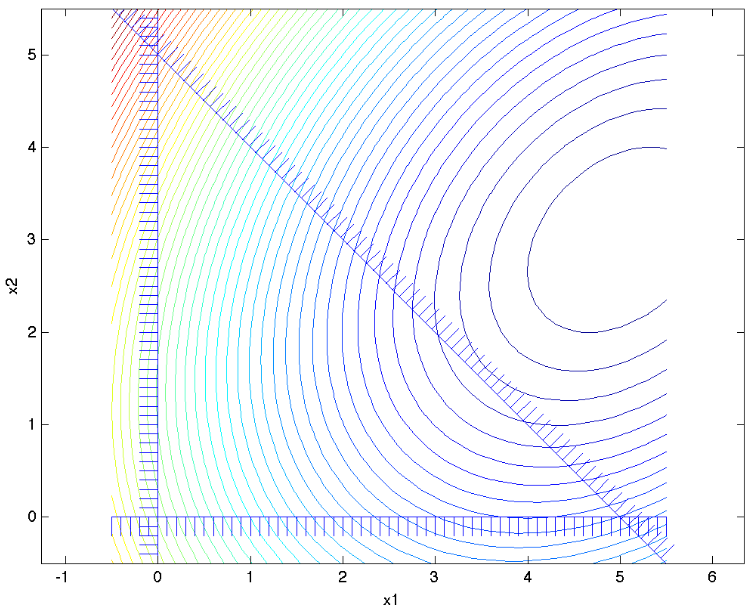


Figure for Exercise 3b: