SF2822 Applied nonlinear optimization, final exam
Thursday May 30 2024 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear programming problem

$$(NLP) \quad \begin{array}{ll} \text{minimize} & (x_1 + 2x_2 + x_3 + 5)^4 + (2x_1 + x_3 - 4)^2 \\ \text{subject to} & -x_1 x_2 \geq -4, \\ & -x_1^2 - 2x_3^2 = -2, \\ & x_1 \geq -1, \\ & x_2 \geq 0. \end{array}$$

A GAMS model of the problem has been created. The GAMS input file can be found at the end of the exam, and a partial GAMS output file reads:

```
               S O L V E      S U M M A R Y

     MODEL   nlpmodel            OBJECTIVE  obj
     TYPE    NLP                 DIRECTION  MINIMIZE
     SOLVER  SNOPT               FROM LINE  26

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      2 Locally Optimal
**** OBJECTIVE VALUE              162.5591

 RESOURCE USAGE, LIMIT          0.003 10000000000.000
 ITERATION COUNT, LIMIT         4    2147483647
 EVALUATION ERRORS              0             0


                       LOWER     LEVEL     UPPER    MARGINAL

---- EQU objfun         -INF       .         .       -1.000
---- EQU cons1         -4.000      .        +INF       .
---- EQU cons2         -2.000    -2.000    -2.000    45.752

                       LOWER     LEVEL     UPPER    MARGINAL

---- VAR obj            -INF    162.559     +INF       .
```

```
---- VAR x

        LOWER      LEVEL      UPPER     MARGINAL

j1     -1.000     -1.000      +INF       24.488
j2        .          .        +INF      285.643
j3      -INF      -0.707      +INF         .
```

(a) Use the GAMS output file to give a point x^* and Lagrange multiplier vector λ^* that together satisfy the first-order necessary optimality conditions for (NLP). .... (4p)

(b) Is x^* a global minimizer to (NLP)? .... (4p)

(c) Assume that for a small positive parameter t, you may either change the second constraint from $-x_1^2 - 2x_3^2 = -2$ to $-x_1^2 - 2x_3^2 = -2 - t$ or the third constraint from $x_1 \geq -1$ to $x_1 \geq -1-t$. Which choice would you make in order to improve the optimal optimal value of the corresponding nonlinear program as much as possible? .... (2p)

2. Consider the quadratic programming problem (QP) defined as

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{3}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & x_1 + x_2 \geq \frac{8}{3}. \end{array}$$

For a positive barrier parameter μ, consider the barrier transformed problem (QP_μ), given by

$$(QP_\mu) \qquad \text{minimize} \quad \tfrac{3}{2}x_1^2 + \tfrac{1}{2}x_2^2 - \mu \ln(x_1 + x_2 - \tfrac{8}{3}),$$

with the associated implicit constraint $x_1 + x_2 > \frac{8}{3}$.

(a) Formulate the primal-dual system of nonlinear equations associated with a primal-dual interior point method for solving (QP). You may assume that the initial point $x^{(0)}$ is strictly feasible with respect to the implict constraint, so that no slack variable is required. Use these equations to find the optimal solution $x(\mu)$ and the corresponding Lagrange multiplier $\lambda(\mu)$ of the barrier-transformed problem (QP_μ). It is possible to obtain analytical expressions for this small problem. .... (5p)

(b) Formulate the system of linear equations to be solved in the first iteration of a primal-dual interior point method associated with (QP). You may again assume that the initial point $x^{(0)}$ is strictly feasible with respect to the implicit constraint, so that no slack variable is required. Formulate the general form, and then insert the specific values $x^{(0)} = (4\ 0)^T$, $\lambda^{(0)} = 2$, for initial barrier parameter $\mu^{(0)} = 1$. .... (5p)

3. Consider the nonlinear programming problem (NLP) defined by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1+x_2)^2 + \frac{3}{2}x_1 - \frac{9}{2}x_2 \\ \text{subject to} & x_1 \cdot x_2 - 1 \geq 0. \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{array}$$

We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)} = (2 \;\; \frac{1}{2})^T$, $\lambda^{(0)} = (1\;0\;0)^T$ and perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. . . . . (10p)

Note: According to the convention of the textbook we define the Lagrangian $\mathcal{L}(x,\lambda)$ as $\mathcal{L}(x,\lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ is the objective function and $g(x)$ is the constraint function, with the inequality constraints written as $g(x) \geq 0$.

4. Derive the expression for the BFGS quasi-Newton update, stating B_{k+1} for a given B_k, where B_k is symmetric and positive definite. First derive the expression for a strictly convex quadratic function, and then generalize to a general nonlinear function. Finally, state and prove a sufficient condition for B_{k+1} to be positive definite. . . . . (10p)

5. Let V and U_i, $i = 1,\ldots,n$, be given symmetric $m \times m$ matrices.

(a) Formulate the problem of finding $w \in I\!R^n$ such that the difference between the largest and the smallest eigenvalue of the matrix $\sum_{i=1}^n U_i w_i - V$ is minimized as a semidefinite program. . . . . (6p)

Hint: It may be helpful to observe that for a symmetric matrix M, it holds that $M - tI \succeq 0$ if and only if $t \leq \eta_{\min}(M)$, where $\eta_{\min}(M)$ denotes the smallest eigenvalue of M.

(b) Formulate the corresponding dual semidefinite program. . . . . (4p)

Hint: It may be helpful to note that a primal-dual pair (P) and (D) of semidefinite programs may be written as

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \sum_{j=1}^n A_j x_j \succeq B, \end{array}$$

$$(D) \quad \begin{array}{ll} \text{maximize} & \operatorname{trace}(BY) \\ \text{subject to} & \operatorname{trace}(A_j Y) = c_j, \quad j = 1,\ldots,n, \\ & Y = Y^T \succeq 0. \end{array}$$

Good luck!

GAMS file for Question 1:

```
sets
j / j1*j3 /;

variables
obj
x(j) ;

equations
        objfun
        cons1
cons2 ;

objfun .. power(x("j1")+2*x("j2")+x("j3")+5,4)+power(2*x("j1")+x("j3")-4,2) =l= obj;
cons1  .. -x("j1")*x("j2") =g= -4;
cons2  .. -power(x("j1"),2)-2*power(x("j3"),2) =e= -2;

x.lo("j1")=-1;
x.lo("j2") = 0;

model nlpmodel / all /;

solve nlpmodel using nlp minimizing obj;
```