SF2822 Applied nonlinear optimization, final exam
Thursday May 28 2020 8.00–13.00

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Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider a nonlinear programming problem (NLP) defined by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & x_1^2 - 2x_1x_3 + e^{x_2} + x_2x_3 + x_3^2 - x_1 - 2x_2 - x_3 \\ \text{subject to} & -x_1^2 - x_2^2 - x_3^2 + 5 \geq 0, \\ & a^Tx + 2 \geq 0, \end{array}$$

where $a \in I\!R^3$ is a given constant vector. Let $\widetilde{x} = (1\ 0\ 1)^T$.

(a) Determine a such that $\widetilde{x}$ fulfils the first-order necessary optimality conditions for (NLP). .... (6p)

(b) For the value on a which you determined in (1a), determine if $\widetilde{x}$ is a local minimizer to (NLP). .... (4p)

2. Consider the nonlinear program (NLP) given by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & x_1^2 + \frac{1}{2}(x_2 - 1)^2 + x_1x_2 \\ \text{subject to} & \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}x_2^2 \leq 2 \\ & 2x_1 + x_2 = 0. \end{array}$$

Assume that one wants to solve (NLP) by a sequential quadratic programming method for the initial point $x^{(0)} = (-1\ 2)^T$ and $\lambda^{(0)} = (2\ 1)^T$.

(a) Your friend SP claims that there is no need to perform any iterations. He claims that $x^{(0)}$ must be a global minimizer to (NLP), since (NLP) is a convex optimization problem and $\nabla f(x^{(0)}) = 0$. Explain why he is wrong. ...... (2p)

(b) Perform one iteration by sequential quadratic programming for solving (NLP) for the given $x^{(0)}$ and $\lambda^{(0)}$, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, and you do not need to perform any linesearch. .... (8p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

3. Consider the quadratic program (QP) defined by

$$(QP)\qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & \begin{array}{rl} x_1 + x_2 & \geq 6, \\ -x_1 + x_2 & \geq -4, \\ x_1 & \geq 2. \end{array} \end{array}$$

Solve (QP) by an active-set method, with the initial point $x^{(0)}$ given by $x^{(0)} = (6\ 2)^T$ and the constraint $-x_1 + x_2 \geq -4$ in the working set. The equality-constrained quadratic programs that arise need not be solved in a systematic way. They may for example be solved graphically. However, the values of the generated iterates $x^{(k)}$ and corresponding Lagrange multipliers $\lambda^{(k)}$ (*if needed*) should be calculated. (10p)

4. Consider the nonlinear programming problem

$$(P)\qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are continuously differentiable.

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu)\qquad \text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(g_i(x)).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ g_i(x)\lambda_i - \mu &= 0, \quad i = 1, \dots, m, \end{aligned}$$

assuming that $g(x) > 0$ and $\lambda > 0$ is kept implicitly. .... (4p)

(b) Let $x(\mu)$, $\lambda(\mu)$ be a solution to the primal-dual nonlinear equations of (4a) such that $g_i(x(\mu)) > 0$, $i = 1, \dots, m$, and $\lambda(\mu) > 0$. Show that $x(\mu)$ is a global minimizer to (P_μ) if f and $-g_i$, $i = 1, \dots, m$, are convex functions on $I\!R^n$. (2p)

(c) Derive the system of linear equations that results when the primal-dual nonlinear equations of (4a) are solved by Newton's method. .... (4p)

5. Consider the nonlinear optimization problem (NLP) given by

$$(NLP)\qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in F, \end{array}$$

where $f(x) = x_1 + 2x_2 + \frac{x_2^2}{x_1}$ and $F = \{x \in I\!R^2 : 3x_1 + 2x_2 = 1, x_1 > 0\}$.

(a) Show that (NLP) is a convex optimization problem. .... (3p)

(b) Show that (NLP) can be reformulated as a semidefinite program. .... (4p)

(c) Formulate the dual problem of the semidefinite program shown in (5b). . (3p)

Hint 1: Introduce a new variable x_3 plus a constraint $x_3 \geq \frac{x_2^2}{x_1}$.

Hint 2: Let M be a symmetric matrix which is partitioned as

$$M = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}.$$

Assume that $A \succ 0$. Then, $M \succeq 0$ if and only if $C - BA^{-1}B^T \succeq 0$.

Hint 3: Let C and X be a symmetric matrix defined as

$$C = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}, \; X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}.$$

Then $\operatorname{trace}(CX) = c_1x_1 + 2c_2x_2 + c_3x_3$.

Good luck!