Data Driven Modeling

Lecture 1

Håkan Hjalmarsson

KTH - Royal Institute of Technology

hjalmars@kth.se

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Outline

Outline

Introduction

- FEL3201 (8hp) / FEL3202 (12hp)
- Course elements
  - 13 lectures to provide an orientation
  - Q&A follow up the next lecture
  - Recommended reading in the form of lecture notes (continuously updated - feedback welcome!), and L. Ljung: system identification - Theory for the User (available online through KTHB)
  - Weekly homework problems. Peer correction.
  - Project. Groups of 2. Complete system id. problem. Preferrably real data. Optimal with something from your own research. Proposals due to hjalmars@kth.se by June 22. Deadline for reports September 15. 5 min. presentations. Date October TBD.
  - 48h take home exam starting at 9:00. Window: August 29 - September 13. Notify hjalmars@kth.se before August 25. Reminder at 8:30 at the day of the exam.

Introduction

- Course requirements
  - Homeworks: 80% solved
  - Exam: 50% for FEL3201. 65% for FEL3202.
  - Project: Approved report & presentation. Project for FEL3202 expected to be extensive (aim for conference paper).
- Many different areas blend together (Systems & Control theory, Mathematical statistics, Probability theory, Machine learning, Optimization theory,...)

Outline

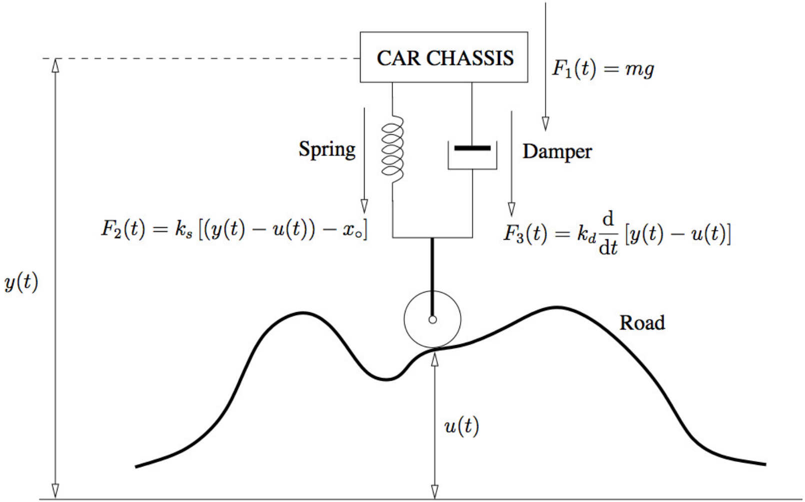
Course Outline

1. Introduction (Friday 15/5, 15-17) . Chapter 1-2 in Lecture Notes (LN). Chapter 1-2 in Ljung.
   - Signals and systems
   - The basic problem
   - Some examples
   - Introduction to parameter estimation
   - Some pitfalls
   - HW: 1.1 a-d (1.1f). 2.1 (2.2, 2.5)) Deadline Tuesday 26/5.
2. Probabilistic models (Tuesday 19/5, 10-12). Chapter 3 in LN. Chapter 4 in Ljung.
   - Models and model structures
   - Estimators
   - A probabilistic toolshed
3. Estimation theory and Wold decomposition (Tuesday 26/5, 10-12). Chapter 4 in LN. Chapter 3 in Ljung
   - Estimation theory
     - Information contents in random variables
     - Estimation of random variables
   - Wold decomposition
4. Unbiased parameter estimation (Friday 29/5, 15-17). Chapter 5 in LN. Chapter 7 in Ljung.
   - The Cramér-Rao lower bound
   - Efficient estimators
   - The maximum likelihood estimator
   - Data compression
   - Uniform minimum variance unbiased estimators
   - Best linear unbiased estimator (BLUE)
   - Using estimation for parameter estimation

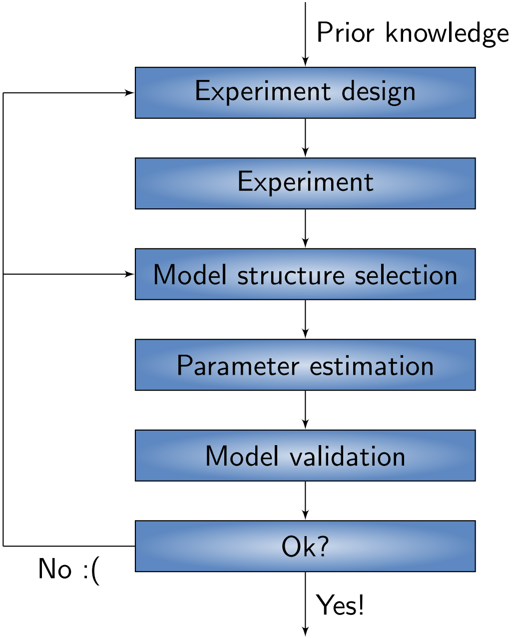
Course Outline

5. Biased parameter estimation (Tuesday 2/6, 10-12) . Chapter 6 in LN.
    - The bias-variance trade-off
    - The Cramér-Rao lower bound
    - Average risk minimization
    - Minimax estimation
    - Pointwise risk minimization
6. Asymptotic theory (Friday 5/6, 15-17). Chapter 7 in L.N. Chapter 8 in Ljung
    - Limits of random variables
    - Large sample properties of estimators
    - Using estimation for parameter estimation, part II
    - Large sample properties of biased estimators
7. Computational aspects (Tuesday 9/6, 08-10). Chapter 10 in Ljung.
    - Gradient based optimization
    - Convex relaxations
    - Integration by Markov Chain Monte Carlo (MCMC) methods
8. Case studies I (Friday 12/6, 10-12)
    - Parametric LTI models
    - Impulse response models
9. Case studies II (Tuesday 16/6, 10-12)
    - Uncertain input models
    - Nonlinear stochastic state-space models
10. Model accuracy (Friday 19/6, 15-17) Chapter 9 in Ljung.
11. Model structure selection and model validation (Tuesday 23/6, 10-12). Chapter 16 in Ljung
12. Experiment design (Tuesday 25/8, 10-12) . Chapter 13 in Ljung.
13. Continuous time identification (Friday 28/8, 15-17)

Introductory example: Shock absorber



System identification, an iterative procedure



Outline

Continuous time signals

Definition

The space $L_p(C)$, $0 < p < \infty$ *consists of all measurable functions* $F : C \to \mathbb{C}^{n\times m}$ *on* C *for which*

$$\|F\|_p := \left(\int_C \|F(t)\|_F^p dt \right)^{1/p} < \infty$$

The class $L_\infty(C)$ *consists of all measurable functions* $F : C \to \mathbb{C}^{n\times m}$ *on* C *for which*

$$\|F\|_\infty := \operatorname*{ess\,sup}_{t\in C} \overline{\sigma}(F(t)) < \infty$$

where $\overline{\sigma}(A)$ *denotes the largest singular value of the matrix* A*.*

The essential supremum for a real-valued function f is defined as

$$\operatorname*{ess\,sup}_{t\in C} f(t) = \inf\{a : \ f(t) \leq a \text{ almost everywhere (a.e.) in } C$$

Continuous time signals

Fourier transform and its inverse

$$S(i\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt, \quad \bar{s}(t) \qquad = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(i\omega) e^{i\omega t} d\omega$$

Theorem

i) *Suppose that $s \in L_1(\mathbb{R})$, then its Fourier transform S is uniformly continuous and vanishes at infinity.*

ii) *Suppose that $s \in L_1(\mathbb{R})$ and that its Fourier transform $S \in L_1(\mathbb{R})$.*

$$\text{Then } \bar{s}(t) = \int_{-\infty}^{\infty} S(i\omega) e^{i\omega t} d\omega$$

is continuous, vanishes at infinity and $\bar{s}(t) = s(t)$ a.e.

iii) *Suppose that $s \in L_p(\mathbb{R})$, $1 < p < \infty$, with Fourier transform S.*

$$\text{Then } \lim_{R \to \infty} \int_{|\omega| \le R} S(i\omega) e^{i\omega t} d\omega = s(t) \quad \text{a.e.}$$

Outline

Discrete time signals

Definition

The class ℓ_p, $0 < p < \infty$, *consists of all sequences* $\{s(t)\}$ *for which*

$$\|s\|_p := \left(\sum_k |s(t)|^p\right)^{1/p} < \infty$$

The class ℓ_∞ *consists of all sequences* $\{s(t)\}$ *for which*

$$\|s\|_\infty := \sup_t |s(t)| < \infty$$

$\ell_p \subset \ell_q$ for $1 \leq p < q \leq \infty$.

$s \in \ell_1 \Rightarrow$ Discrete Time Fourier transform (Fourier series)

$$S(e^{i\omega}) = \sum_{t=-\infty}^{\infty} s(t)e^{-i\omega t}$$

$$S \in L_1(\mathbb{T}),\ \mathbb{T} = \{|z| = 1\} \Rightarrow \bar{s}(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{i\omega})e^{i\omega t}d\omega = s(t)$$

Discrete time signals

ℓ_2 and $L_2(\mathbb{T})$ Hilbert spaces with inner products

$$\langle s, v\rangle = \sum_t \mathrm{Trace}\left\{v^*(t)s(t)\right\}, \ \langle S, V\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Trace}\left\{V^*(e^{i\omega})S(e^{i\omega})\right\} d\omega$$

$b_k(\omega) = e^{i\omega k}$, complete orthonormal system for $L_2(\mathbb{T}) \Rightarrow$

Theorem

Any $S \in L_2(\mathbb{T})$ can be represented as $S(e^{i\omega}) = \sum_{t=-\infty}^{\infty} s(t)e^{-i\omega t}$ where

$$s(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{i\omega})e^{i\omega t} d\omega$$

What does $S = 0$ mean in $L_2(\mathbb{T})$? $\|S\|_2 = 0$. Equivalence classes.
ℓ_2 and $L_2(\mathbb{T})$ isomporphic: 1-1 relationship between elements.

Geometric properties preserved: $\langle S, V\rangle = \langle s, v\rangle$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |S(e^{i\omega})|^2 d\omega = \|S\|_2^2 = \|s\|_2^2 = \sum_{t=-\infty}^{\infty} |s(t)|^2$$

Discrete time signals

z-transform: $S(z) := \sum_{k=-\infty}^{\infty} s(k)z^{-k}$ (Laurent series)
Holomorphic (analytic) in an annulus centered at the origin.

Definition

H_p, $0 < p < \infty$ is the class of functions $F : \mathbb{T} \to \mathbb{C}^{n\times m}$ for which all elements are holomorphic in $|z| > 1$ and for which there is an $M < \infty$ such that

$$\int_{-\pi}^{\pi} \|F(re^{\omega})\|_F^p d\omega \leq M, \quad 1 < r < \infty$$

Theorem (H_p vs $L_p(\mathbb{T})$)

Let $1 < p < \infty$. $S \in H_p \Leftrightarrow S(z) = \sum_{t=0}^{\infty} \bar{s}(t)z^{-t}$ where $\{\bar{s}(t)\}_{t=1}^{\infty}$ are the Fourier coefficiencts of some function in $L_p(\mathbb{T})$.

Dynamic systems

Linear time-invariant (LTI)

$$y(t) = \sum_{k=-\infty}^{\infty} g(k)u(t-k),$$

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Bounded-Input-Bounded-Output (BIBO) stability: $g \in \ell_1$

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$$\|G\| = \sup_u \frac{\|Gu\|_\infty}{\|u\|_\infty} = \|g\|_1$$

$$\|G\| = \sup_u \frac{\|Gu\|_2}{\|u\|_2} = \sup_\omega |G(e^{i\omega})|$$

Dynamic systems

- Linear state space description

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t) + K(\theta)e(t)$$
$$y(t) = C(\theta)x(t) + D(\theta)u(t) + e(t)$$

  - $\{e(t)\}$ noise/disturbance
  - θ vector of unknown parameters
  - Black-box or (semi-)physical (grey-box)

Dynamic systems

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\end{aligned}
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  - $\{e(t)\}$ noise/disturbance
  - θ vector of unknown parameters
  - Black-box or (semi-)physical (grey-box)

- Non-linear

$$
\begin{aligned}
x(t+1) &= f(x(t), u(t), w(t), \theta) \\
y(t) &= h(x(t), u(t), e(t), \theta)
\end{aligned}
$$

Common linear black-box structures

- FIR

$$
\begin{aligned}
y(t) &= b_1 u(t-1) + \ldots b_n u(t-n) + e(t) \\
&= \begin{bmatrix} u(t-1) & \ldots & u(t-n) \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + e(t) = \varphi^T(t)\theta + e(t)
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Compact form:

$y(t) = B(q)u(t) + e(t) = (b_1 q^{-1} + \ldots + b_n q^{-n})u(t) + e(t).$

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- General:

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t)$$

where G and H are rational discrete-time transfer functions.

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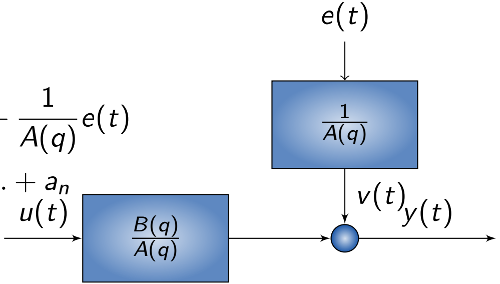
Common linear black-box structures

- General: $y(t) = G(q,\theta)u(t) + H(q,\theta)e(t)$

- ARX

$$y(t) = \frac{B(q)}{A(q)}u(t) + \frac{1}{A(q)}e(t)$$

$$A(q) = 1 + a_1 + \ldots + a_n$$



Common linear black-box structures

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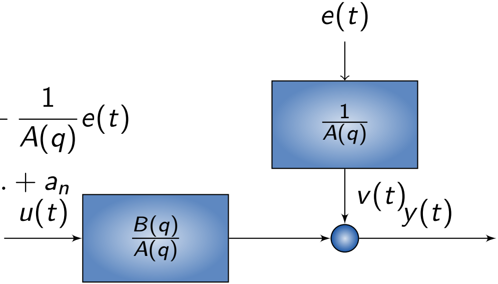

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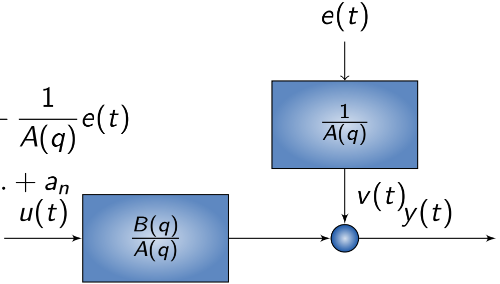


Can be written $A(q)y(t) = B(q)u(t) + e(t)$
which is equivalent to

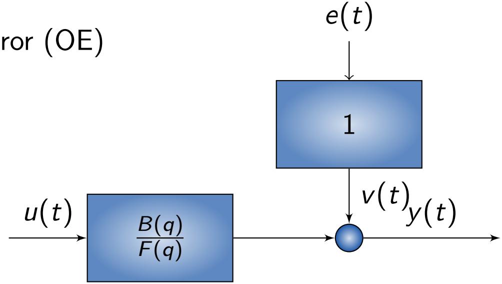
$$y(t) = \varphi^T\theta + e(t)$$

$$\varphi(t) = \begin{bmatrix} -y(t-1) & \ldots - y(t-n) & u(t-1) & \ldots & u(t-n) \end{bmatrix}^T$$

$$\theta = \begin{bmatrix} a_1 & \ldots & a_n & b_1 & \ldots & b_n \end{bmatrix}^T$$

Common linear black-box structures

- Output-Error (OE)



Common linear black-box structures

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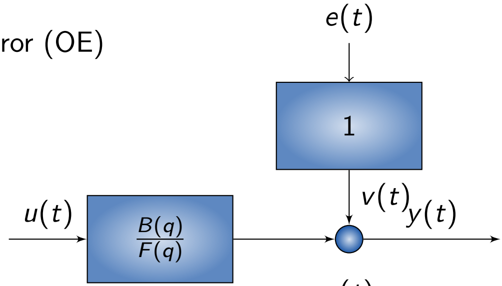
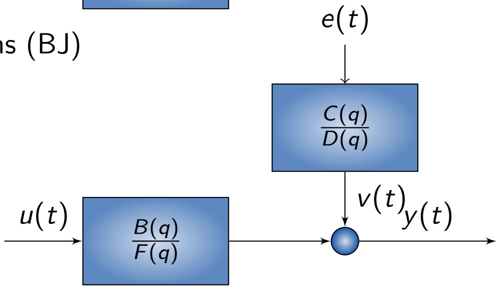


- Box-Jenkins (BJ)



Continuous time models

$$\dot{x}(t) = \mathcal{A}(\theta)x(t) + \mathcal{B}(\theta)u(t) + w(t)$$
$$y(t) = \mathcal{C}(\theta)x(t) + \mathcal{D}(\theta)u(t) + v(t)$$

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Sampling gives

$$x(t+1) \approx A(\theta)x(t) + B(\theta)u(t) + K(\theta)e(t)$$
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Important to use correct intersample behaviour of input.

Common nonlinear black-box models

- Predictor models

$$y(t) = g(\varphi(t), \theta) + e(t)$$

where $\varphi(t)$ (nonlinear transformations of) past inputs and outputs.

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  - Radial basis functions

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- NLARX: $\varphi(t)$ past inputs and outputs
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-
-

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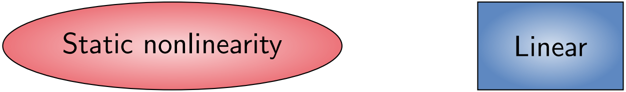
  -
  -
  -

- Block oriented models

Block-oriented models



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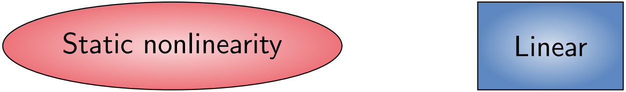
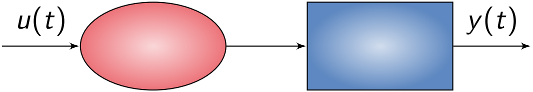


- Hammerstein (nonlinear actuator)



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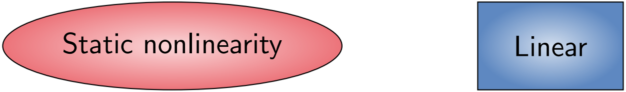

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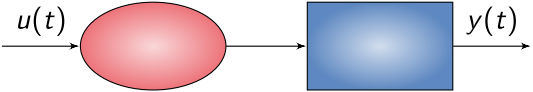
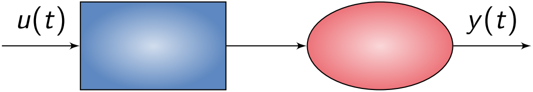


- Wiener (nonlinear sensor)



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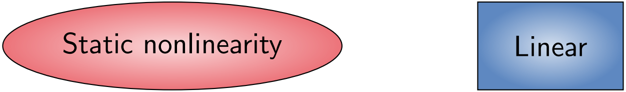

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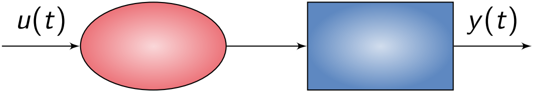

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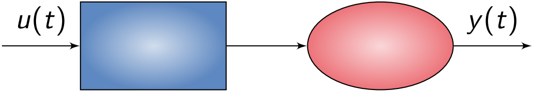
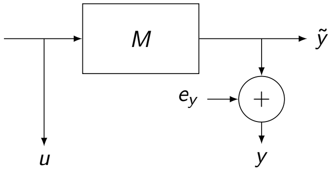


- Hammerstein-Wiener

Outline

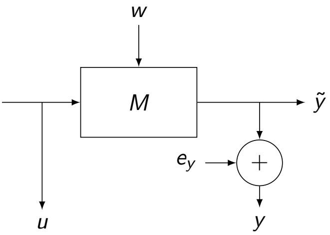
Example 1: Scalar LTI model



$$\mathbf{y} = \Phi \mathbf{g} + \mathbf{e}_y$$

- Measurements: $\mathbf{y} \in \mathbb{R}^N$ (u known exactly and can be considered part of the model)
- Unknowns: $\mathbf{g} \in \mathbb{R}^n$, $\mathbf{e}_y \in \mathbb{R}^N$

Example 2: Scalar LTI state-space model

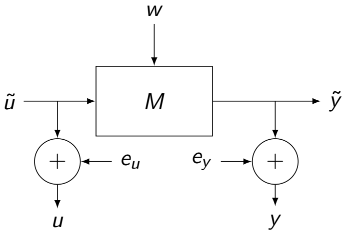


$$\mathbf{x} = F(\boldsymbol{\theta})\mathbf{u} + G(\boldsymbol{\theta})\mathbf{w}$$

$$\mathbf{y} = H(\boldsymbol{\theta})\mathbf{x} + \mathbf{e}_y, \quad \mathbf{y} \in \mathbb{R}^N$$

- Measurements: $\mathbf{y} \in \mathbb{R}^N$
- Unknowns: $\mathbf{w} \in \mathbb{R}^{mN}$, $\boldsymbol{\theta} \in \mathbb{R}^{m^2+2m}$, $\mathbf{e}_y \in \mathbb{R}^N$

Example 3: Scalar LTI state-space EIV model



$$\mathbf{x} = F(\boldsymbol{\theta})\mathbf{u} + G(\boldsymbol{\theta})\mathbf{w}$$
$$\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{e}_u$$
$$\mathbf{y} = H(\boldsymbol{\theta})\mathbf{x} + \mathbf{e}_y$$

- Model order: m
- Measurements: $\mathbf{u} \in \mathbb{R}^N$, $\mathbf{y} \in \mathbb{R}^N$
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Outline

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Notice that the ranking function has nothing to do with the data. The only connection to the data is that we maximize over the unknowns consistent with the data.

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From here on it can only become more confusing

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$$\mathbf{y}(\mathbf{g}, \mathbf{e}_y) = \Phi\mathbf{g} + \mathbf{e}_y, \quad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{g} \\ \mathbf{e}_y \end{bmatrix}$$

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$$p(\boldsymbol{\xi}) = \mathcal{N}(\mathbf{e}_y; 0, \lambda_{e_y} I)\, \mathcal{N}(\mathbf{g}, 0, K_g)$$

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Special case: $\mathbf{y} = \mathbf{g} + \mathbf{e}_y$ $(\Phi = I)$, $K_g = \lambda_g I$

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Average over fs that are unfalsified

Outline

Choosing the ranking function $p(\boldsymbol{\xi})$

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Maximize the average of the rankings

Example 1 cont'd: Special case

$$\mathbf{y}(\mathbf{g}, \mathbf{e}_y) = \mathbf{g} + \mathbf{e}_y, \quad \mathbf{y} \in \mathbb{R}^N$$

$$\mathbf{e}_y \sim \mathcal{N}(0, \lambda_{e_y} I), \ \mathbf{g} \sim \mathcal{N}(\bar{\mathbf{g}}, \lambda_g I)$$

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Estimate

$$\hat{\lambda}_g = \frac{1}{N} \|\mathbf{y} - \bar{\mathbf{g}}\|^2 - \lambda_{e_y}$$

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Spread of $\mathbf{y}$ around $\bar{\mathbf{g}}$, accounting for spread of $\mathbf{e}_y$.

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Spread of $\mathbf{y}$ around $\bar{\mathbf{g}}$, accounting for spread of $\mathbf{e}_y$.

$$\hat{\mathbf{g}}(\hat{\lambda}_g) = \frac{\hat{\lambda}_g}{\hat{\lambda}_g + \lambda_{e_y}} \mathbf{y}$$

Example 1 cont'd: Special case

$$\mathbf{y}(\mathbf{g}, \mathbf{e}_y) = \mathbf{g} + \mathbf{e}_y, \quad \mathbf{y} \in \mathbb{R}^N$$

$$\mathbf{e}_y \sim \mathcal{N}(0, \lambda_{e_y} I), \ \mathbf{g} \sim \mathcal{N}(\bar{\mathbf{g}}, \lambda_g I)$$

$$-\log p(\mathbf{y}; \lambda_g) = \frac{\|\mathbf{y} - \bar{\mathbf{g}}\|^2}{\lambda_g + \lambda_{e_y}} + N \log(\lambda_g + \lambda_{e_y})$$

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- Hypothesis violated (T large) $\Rightarrow$ Data used

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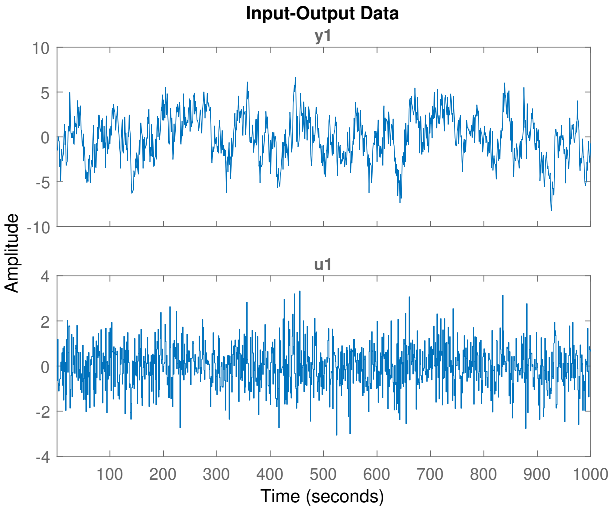
Model simulation

Model

$$y(t) = \frac{bq^{-1}}{1 + fq^{-1}} u(t)$$

Data



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b and f determined by minimizing

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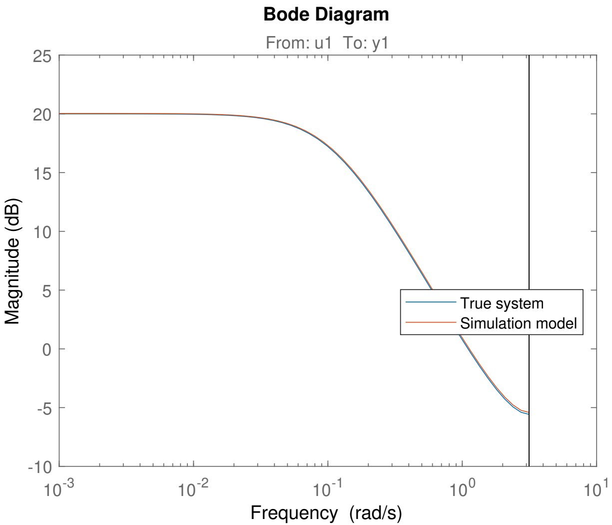
that is

$$\hat{y}(t; b, f) = -f\hat{y}(t-1, b, f) + bu(t-1)$$
$$\hat{y}(1; b, f) = 0$$
$$\vdots \qquad \vdots$$
$$\hat{y}(5; b, f) = -f^3 bu(1) + f^2 bu(2) - fbu(3) + bu(4)$$
$$\vdots \qquad \vdots$$

Model simulation



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Least-squares problem!!!

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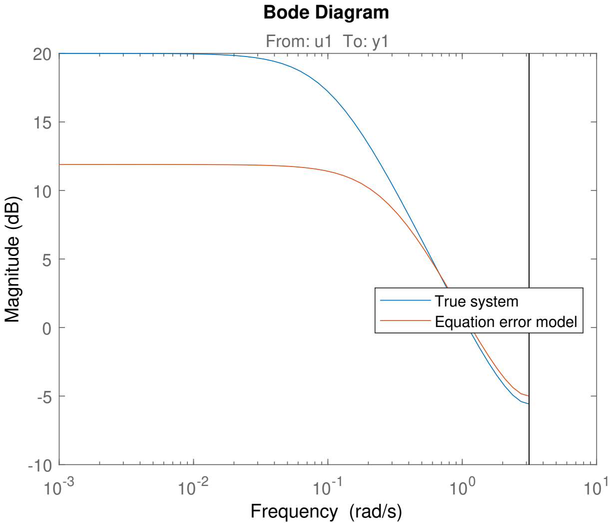
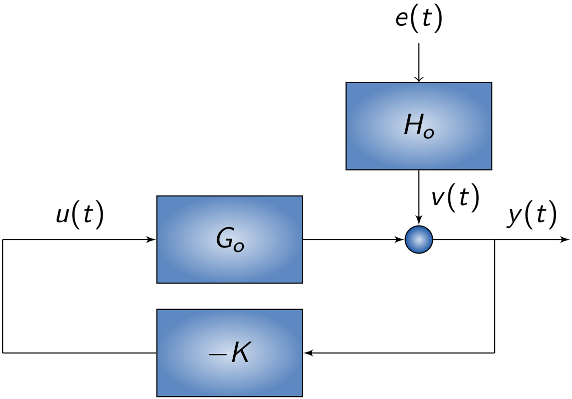
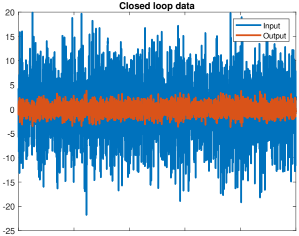


Why different results. Which one to use?

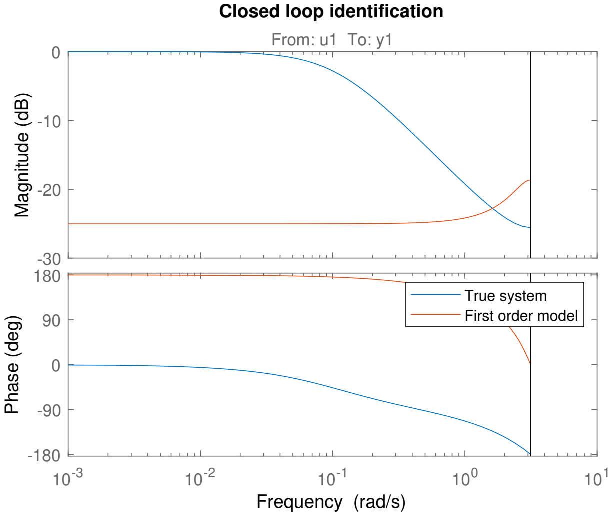
Closed loop identification





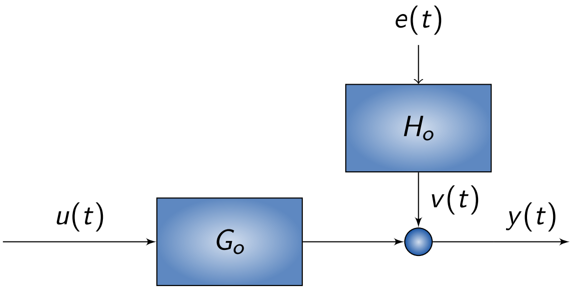
Closed loop identification

Result:

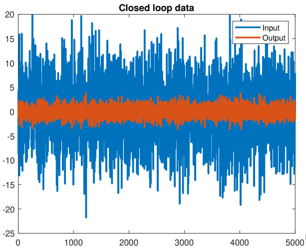


Closed loop identification

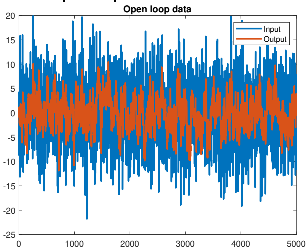
Open loop identification



Data same characteristics as in closed loop experiment:



0.5cm



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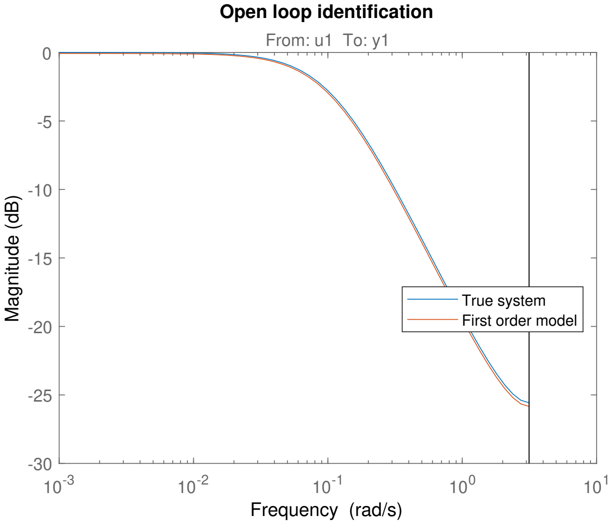
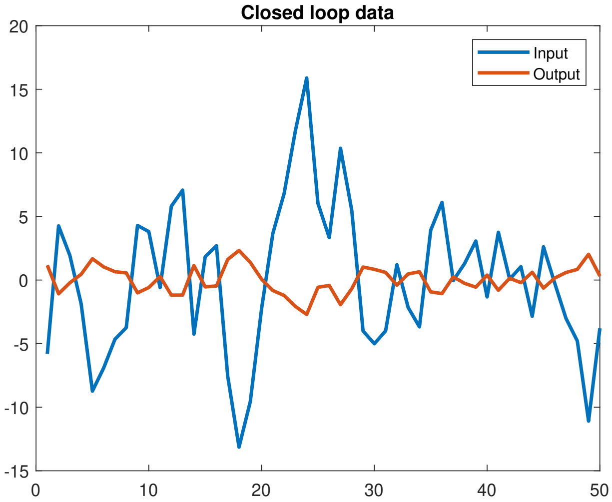


What so peculiar about closed loop identification?

Closed loop identification

Close up

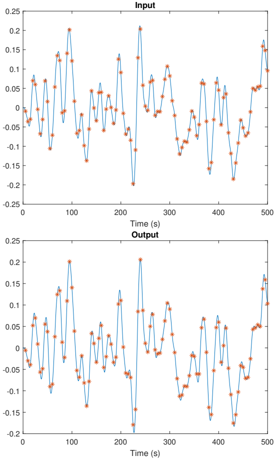


Opposite response to the eye!

Sampling

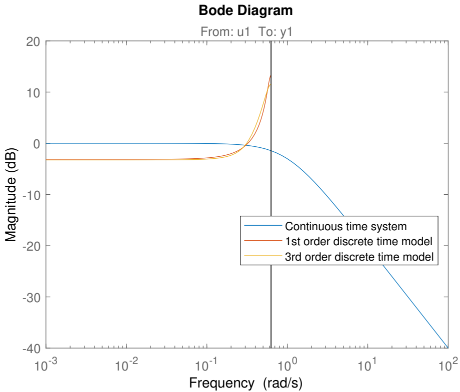
$$G(s) = \frac{1}{s+1}$$

Data:



Sampling

$$y(nT) = \frac{\sum_{k=1}^{n} b_k q^{-k}}{1 + \sum_{k=1}^{n} f_k q^{-k}} u(nT)$$



Noise free data, fast sampling. Yet problem???

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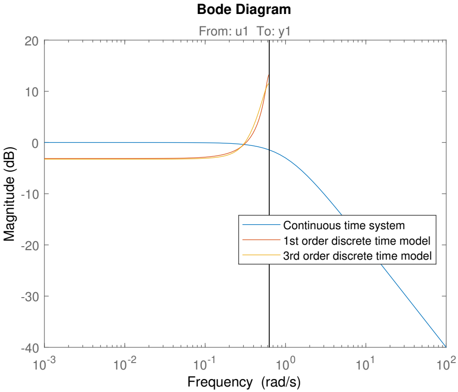
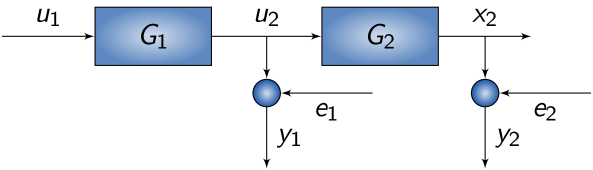
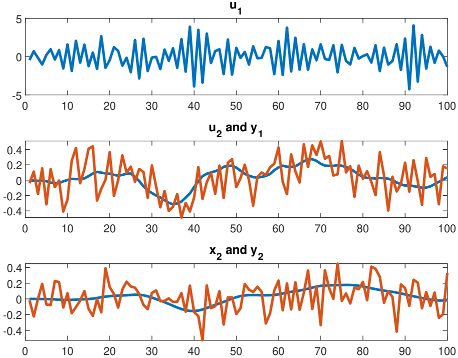


Measurement errors

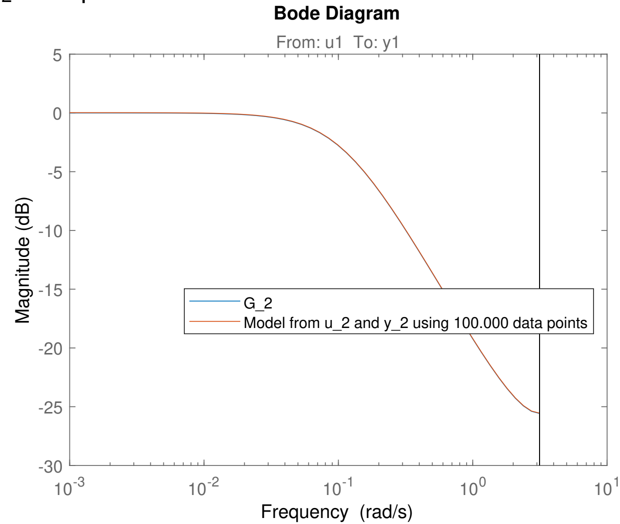


Interested in G_2 but also G_1 (high order) unknown
Large data set (100.000 samples). First 1000 shown



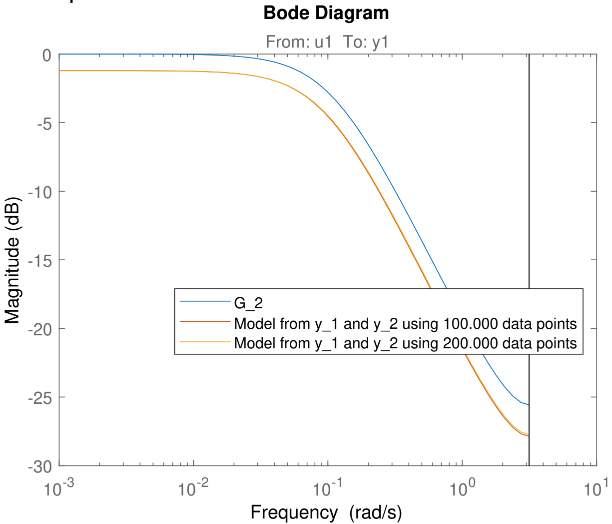
Measurement errors

Using u_2 as input



Measurement errors

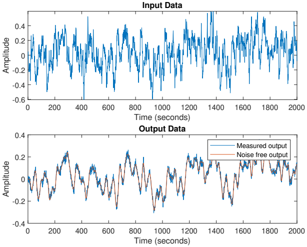
Using y_1 as input in the model



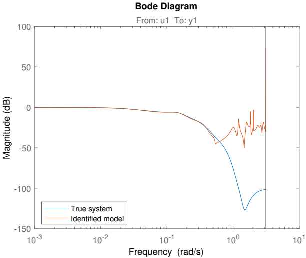
How handle measurement errors on inputs?

Complex models

System of known order 25

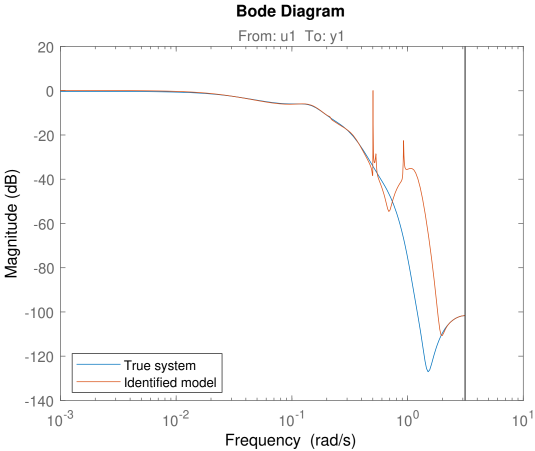


State-of-the art:



Complex models

Recall: Highly non-linear optimization problem. Need good inital values. Let us start at true values.



Still problems. How to deal with complex systems?

Hilbert spaces

Let $\mathcal{V}$ be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
2. $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$
3. $\langle u, v \rangle = \langle v, u \rangle^*$
4. $\langle v, v \rangle \geq 0$ with equality iff $v = 0$

Norm: $\|v\| = \sqrt{\langle v, v \rangle}$

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$$\langle u, v \rangle = M, \quad M_{i,j} = \langle u_i, v_j \rangle$$

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$$\langle X, Y \rangle = X^T Y$$

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Example 2: Let $\mathbf{x} \in \mathbb{R}^{n_x}$ and $\mathbf{y} \in \mathbb{R}^{n_y}$ be random vectors with finite second moments. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbb{E}\left\{ \mathbf{x}\mathbf{y}^T \right\}$$

Orthogonal projections

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Let $u \in \mathcal{H}$ be given and let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace to $\mathcal{H}$. Then there exists a unique $v \in \mathcal{S}$ such that $u - v \perp \mathcal{S}$. The element v is the unique solution to

$$\min_{v \in \mathcal{S}} \|u - v\|$$

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It follows that $u \in \mathcal{H}$ has a unique decomposition

$u = u_{\mathcal{S}} + u_{\mathcal{S}^\perp}$, where $u_{\mathcal{S}^\perp} = u - u_{\mathcal{S}} \in \mathcal{S}^\perp$ (subspace orthogonal to $\mathcal{S}$)

Orthogonal projections: Pythagoras relation

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Matrix inequality

Note: Projection $u_{\mathcal{S}}$ has smaller "norm" than u: $\langle u, u\rangle - \langle u_{\mathcal{S}}, u_{\mathcal{S}}\rangle \geq 0$

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$$U_{\mathcal{S}} = U^T Y (Y^T Y)^{-1} Y$$
$$0 \geq (U - U_{\mathcal{S}})^T (U - U_{\mathcal{S}}) = U^T U - U^T Y (Y^T Y)^{-1} Y^T U$$

Summary