MATRIX ALGEBRA
MAGNUS JANSSON
Deadline: 2020–03–26, 13.00

Homework #1

Read Chapter 0 in "Matrix Analysis" and learn as much as possible!

1. Determine the range- and the null-spaces of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

   What are the dimensions of these spaces? What is the rank of A?

2. Let $A \in M_{m,n}(\mathbf{F})$ and $B \in M_{p,n}(\mathbf{F})$. Prove that

$$\text{nullspace}(A) \cap \text{nullspace}(B) = \text{nullspace} \begin{bmatrix} A \\ B \end{bmatrix}$$

3. Let $A = [a_{ij}] \in M_{m,n}(\mathbf{C})$ and $B \in M_{n,m}(\mathbf{C})$. Show that $\text{tr}(AB) = \text{tr}(BA)$ and that $\text{tr}(AA^*) = \sum_{ij} |a_{ij}|^2$ (the squared Frobenius norm).

4. Show that $\det(I + AB) = \det(I + BA)$ where A and B may be rectangular matrices of appropriate dimensions. (Hint: You may e.g. use the Schur complement determinantal formulae.)

5. Verify the statement that y_2 is orthogonal to z_1 in Section 0.6.4 Gram-Schmidt orthonormalization. Make a graph illustrating the first step of the procedure. Use the fact that $\langle y_2, y_2 \rangle \geq 0$ to prove the Cauchy-Schwarz inequality (in terms of x_2 and y_1). (There is a typo in the book $v_1, \ldots, v_n$ should be $x_1, \ldots, x_n$.)

6. Prove the "push through rule:"

$$A(I_m + BA)^{-1} = (I_n + AB)^{-1}A$$

   where inverses are assumed to exist, I_n is an $n \times n$ identity matrix, $A \in M_{n,m}(\mathbf{F})$ and $B \in M_{m,n}(\mathbf{F})$.

7. Let $S \in M_n(\mathbf{R})$ be a skew-symmetric matrix. First prove that $I - S$ is nonsingular. Then, if $A = (I + S)(I - S)^{-1}$, show that $A^{-1} = A^T$ if the inverse exists.