KTH ROYAL INSTITUTE OF TECHNOLOGY

Lecture 1 : Chapter 0 Review and Miscellanea

Vector spaces

A set V is a vector space over a field $\mathbf{F}$ (for example, the field of real $\mathbf{R}$ or complex numbers $\mathbf{C}$) if, given

- an operation *vector addition* defined in V, denoted $v + w$ (where $v, w \in V$), and
- an operation *scalar multiplication* in V, denoted $a * v$ (where $v \in V$ and $a \in \mathbf{F}$),

the following ten properties hold for all $a, b \in \mathbf{F}$ and u, v, and $w \in V$:

Vector spaces cont'd

1. $v + w$ belongs to V. (Closure of V under vector addition.)
2. $u + (v + w) = (u + v) + w$. (Associativity of vector addition in V.)
3. There exists a neutral element 0 in V, such that for all elements v in V, $v + 0 = v$. (Existence of an additive identity element in V.)
4. For all v in V, there exists an element w in V, such that $v + w = 0$. (Existence of additive inverses in V.)
5. $v + w = w + v$. (Commutativity of vector addition in V.)

Vector spaces cont'd

6. $a * v$ belongs to V. (Closure of V under scalar multiplication.)
7. $a * (b * v) = (ab) * v$. (Associativity of scalar multiplication in V.)
8. If 1 denotes the multiplicative identity of the field **F**, then $1 * v = v$. (Neutrality of one.)
9. $a * (v + w) = a * v + a * w$. (Distributivity with respect to vector addition.)
10. $(a + b) * v = a * v + b * v$. (Distributivity with respect to field addition.)

Vector spaces, cont'd

The concept of a vector space is entirely abstract. To determine if a set V is a vector space, one only has to specify the set V, a field $\mathbf{F}$, and define vector addition and scalar multiplication in V. Then, if V satisfies the above ten properties, it is a vector space over the field $\mathbf{F}$.

The members of a vector space are called vectors.

Examples

We will typically encounter vector spaces formed by n-tuples of scalars from $\mathbf{F}$ denoted $\mathbf{F}^n$. (E.g., $\mathbf{R}^n$ and $\mathbf{C}^n$.)

Note however that vector spaces are also generated by, e.g.,
(i) polynomials with coefficients from $\mathbf{F}$
(ii) or functions over an interval $[a, b] \subset \mathbf{R}$.

Some other examples are:

- $\mathbf{C}$ is a vector space over $\mathbf{R}$
- $\mathbf{R}$ is a vector space over the rational numbers

Subspaces and Span

A subspace of a vector space V is a subset of V that is by itself a vector space.

Examples: $\{[\alpha\ 2\alpha]^T : \alpha \in \mathbf{R}\}$ is a subspace of $\mathbf{R}^2$.
and, similarly, $\{\alpha + j2\alpha : \alpha \in \mathbf{R}\}$ is subspace of the vector space $\mathbf{C}$ over the field $\mathbf{R}$.

Let S be a subset of V then
$\text{span}(S) = \{\sum_i a_i v_i : a_i \in \mathbf{F}, v_i \in S\}$. Note that $\text{span}(S)$ is always a subspace even if S may not be.

A set S is said to span V if $\text{span}(S) = V$.

Sum and Direct sum

The sum of two subspaces S_1 and S_2 is the subspace:

$$S_1 + S_2 = \text{span}\{S_1 \cup S_2\} = \{x + y \ : \ x \in S_1, \ y \in S_2\}$$

If $S_1 \cap S_2 = \{0\}$, we say that the sum is a *direct sum*

$$S_1 \oplus S_2$$

Every $z \in S_1 \oplus S_2$ can be *uniquely* written as $z = x + y$ with $x \in S_1$ and $y \in S_2$.

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- A subset S of the vector space V is said to span V if every element of V can be represented as a linear combination of elements from S.
- A *linearly independent* set spanning V is a *basis* for V.

Basis cont'd

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- The "standard basis" of $\mathbf{R}^n$ (or $\mathbf{C}^n$) is $\{e_1, ..., e_n\}$ where $e_1 = [1\ 0\ 0...]^T$ etc.

Isomorphism

Let U and V be vector spaces over $\mathbf{F}$ and let $f : U \to V$ be an *invertible* function such that
$f(ax + by) = af(x) + bf(y);\ \forall x, y \in U$ and $a, b \in \mathbf{F}$.

Then f is said to be an isomorphism and U and V are isomorphic.

If U and V are finite dimensional then they are isomorphic iff they have the same dimension. This implies that all n-dim real vector spaces are isomorphic to $\mathbf{R}^n$.

Example

Consider the vector space V generated by nth order real polynomials with basis $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$.

All elements $p \in V$ can be represented uniquely by $p = \sum_i a_i x^i$ with $a_i \in \mathbf{R}$ and hence we can associate p with $[p]_\mathcal{B} = [a_0, a_1, \dots, a_n]^T$.

The mapping $p \to [p]_\mathcal{B}$ is an isomorphism between V and $\mathbf{R}^{n+1}$ for any basis $\mathcal{B}$.

Matrices

A Matrix: “Array of scalars” or “linear transformation between two vector spaces”

Notation: $A \in M_{m,n}(\mathbf{F})$. Simplifications: $M_{n,n}(\cdot) = M_n(\cdot)$ often $M_{m,n}(\mathbf{C}) = M_{m,n}$.

Linear transformation

Let U (n-dim) and V (m-dim) be vector spaces over $\mathbf{F}$. Further let $\mathcal{B}_U$ and $\mathcal{B}_V$ be bases and let the vectors in U and V be represented by their n- and m-tuples over $\mathbf{F}$.

A linear transformation is a function $T : U \to V$ such that $T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$ for all $a_i \in \mathbf{F}$ and $x_i \in U$.

The linear transformation $y = T(x)$ can be represented by a matrix $A \in M_{m,n}(\mathbf{F})$ as follows: $[y]_{\mathcal{B}_V} = A[x]_{\mathcal{B}_U}$.

Note that the matrix representation depends on the bases!

Range and null-space

With no loss of generality we will think of $A \in M_{m,n}(\mathbf{F})$ as a linear transformation from $\mathbf{F}^n$ to $\mathbf{F}^m$.

The *domain* is $\mathbf{F}^n$ and the *range* is $\{y \in \mathbf{F}^m : y = Ax,\ x \in \mathbf{F}^n\}$.

The *null-space* of A is $\{x \in \mathbf{F}^n : Ax = 0\}$.

It always holds that

$$n = \text{dim null-space of } A + \text{dim range of } A$$

Matrix multiplication and commutation

Matrix multiplication (in the usual way) of $A \in M_{m,n}(\mathbf{F})$ and $B \in M_{p,q}(\mathbf{F})$ is only defined if $p = n$. It corresponds to a composition of linear transformations.

Note that AB do not in general commute; that is, $AB \neq BA$. Special cases exist, but the (scaled) identity matrix is the only matrix that commutes with any other matrix.

Transpose and conjugate transpose

If $A = [a_{ij}] \in M_{m,n}(\mathbf{F})$ then the *transpose* of A, $A^T \in M_{n,m}(\mathbf{F})$, has a_{ij} as its (j, i):th element.

The *conjugate transpose* A^* of $A \in M_{m,n}(\mathbf{C})$ is defined as $A^* = \bar{A}^T$ where $\bar{A}$ is the conjugate of A.

Other names for conjugate transpose are: adjoint, Hermitian adjoint, Hermitian transpose. Often it is also denoted A^H.

Note that $(AB)^T = B^T A^T$.

A matrix is symmetric if $A^T = A$ and skew symmetric if $A^T = -A$.

A matrix is Hermitian if $A^* = A$ and skew Hermitian if $A^* = -A$.

Trace

The trace of $A = [a_{ij}] \in M_{m,n}(\mathbf{F})$ is the sum of the main diagonal elements:

$$\text{tr}(A) = \sum_{i=1}^{q} a_{ii}; \quad q = \min\{m, n\}$$

Determinants

Let $A = [a_{ij}] \in M_n(\mathbf{F})$ and let A_{ij} denote the submatrix obtained by deleting row i and column j of A.
Laplace expansion:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

$\det(a_{ij}) = a_{ij}$

$\det(A) = 0$ iff a subset of its rows (or equiv. columns) is linearly dependent.

Multiplicativity: $\det(AB) = \det(A)\det(B)$

Elementary operations

- Interchange of two rows
- Multiplication of a row by a scalar
- Addition of a scalar multiple of one row to another row

Each $A \in M_{m,n}(\mathbf{F})$ can be reduced to its RREF (row reduced echelon form) by elementary operations: Canonical (unique) form for matrices (theoretically) useful for determining rank, solving linear system of equations, computing determinants.

Rank

$\text{rank}(A) =$ is the largest number of linearly independent columns (or rows) of A.

Linear system of equations:

Note that $Ax = b$ has either $0,\ 1$, or ∞ many solutions x.

If it has solutions, it is called *consistent*. That happens iff $\text{rank}([A\ b]) = \text{rank}(A)$.

Rank cont'd

Characterizations of rank: see book 0.4.4
Rank inequalities: see book 0.4.5
Rank equalities: see book 0.4.6.

Note in particular: If $A \in M_{m,n}(\mathbf{F})$ and $\text{rank}(A) = k$ then it can always be written as

$$A = XBY$$

where $X \in M_{m,k}(\mathbf{F})$, $Y \in M_{k,n}(\mathbf{F})$ are full rank, and $B \in M_{k,k}(\mathbf{F})$ is nonsingular.

Nonsingularity

A linear transformation (or matrix) is said to be nonsingular if it produces the output 0 only for the input 0, otherwise it is singular.

If $A \in M_{m,n}(\mathbf{F})$ and $m < n$ then A is always singular.

$A \in M_n(\mathbf{F})$ is *invertible* if there exists a matrix A^{-1} such that $A^{-1}A = I$; then also $AA^{-1} = I$ and A^{-1} is unique.

Equivalently, $A \in M_n(\mathbf{F})$ is *invertible* if the linear transformation A is one-to-one and the inverse (linear) transformation exists.

Inner product

- Consider elements of $\mathbf{F}^n$ as column vectors ($\mathbf{F}^n = M_{n,1}(\mathbf{F})$).
- Let $x, y \in \mathbf{C}^n$. The scalar $y^*x \equiv \langle x, y\rangle$ is the (standard or usual) inner (scalar) product of x and y on $\mathbf{C}^n$ (there are others).
- We say $x, y \in \mathbf{C}^n$ are *orthogonal* if $\langle x, y\rangle = 0$.
- The *Euclidean length* of $x \in \mathbf{C}^n$ is $\langle x, x\rangle^{1/2}$.

Inner product cont'd

- The Cauchy-Schwartz inequality: $|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$ with equality iff x and y are dependent.
- The angle between two vectors is defined by: $\cos(\theta) = \frac{|\langle x,y\rangle|}{\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}}$
- Gram-Schmidt orthonormalization – orthonormal bases – orthogonal complements

Partitioned matrices

If

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

then the *Schur complement* to A_{11} is $S_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Similarly, $S_{22} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ is the Schur complement of A_{22}.

One way of writing the inverse of A is

$$A^{-1} = \begin{bmatrix} S_{22}^{-1} & -A_{11}^{-1}A_{12}S_{11}^{-1} \\ -S_{11}^{-1}A_{21}A_{11}^{-1} & S_{11}^{-1} \end{bmatrix}$$

“Matrix inversion lemma”

or the Sherman-Morrison-Wodbury formula...

If $B = A + XRY$, then (assuming the inverses exist)

$$B^{-1} = A^{-1} - A^{-1}X(R^{-1} + YA^{-1}X)^{-1}YA^{-1}$$

More topics ...

(Classical) Adjoint of A: $\text{Adj}(A)$ (also called adjugate)

Cramér's rule

Schur complements and determinants

Special matrices :

- Diagonal – triangular etc
- Permutation
- Circulant – Toeplitz – Hankel – Hessenberg – tridiagonal
- Vandermonde

Change of basis