## SF2812 Applied linear optimization, final exam
## Friday March 8 2019 8.00–13.00
## Brief solutions

**1.** (a) Since $\widetilde{x}$ is a feasible solution to $(PLP)$ and $(PLP)$ is a minimization problem, $c^T\widetilde{x}$ is an upper bound for optval$(PLP)$.

(b) By strong duality for linear programming, the optimal values of $(PLP)$ and $(DLP)$ are equal, if both problems are feasible. Therefore, $c^T\widetilde{x}$ is an upper bound for optval$(DLP)$. There is no implication that optval$(DLP) > -\infty$ by existence of primal feasible solution.

(c) No such $\eta$ and $q$ can exist since $(PLP)$ is feasible. To see this, note that premultiplication of $A^T\eta + q = 0$ by $\widetilde{x}^T$ gives

$$0 = \widetilde{x}^T(A^T\eta + q) = b^T\eta + \widetilde{x}^Tq.$$

If $\widetilde{x} \geq 0$ and $q \geq 0$, it follows that $\widetilde{x}^Tq \geq 0$. Hence, if $b^T\eta > 0$, we have

$$b^T\eta + \widetilde{x}^Tq > 0,$$

which is a contradiction.

(If $(DLP)$ has a feasible solution $\widetilde{y}$, $\widetilde{s}$, then $\widetilde{y} + \alpha q$, $\widetilde{s} + \alpha\eta$ is feasible to $(DLP)$ for $\alpha > 0$, and $b^T(\widetilde{y} + \alpha\eta) \to \infty$ as $\alpha \to \infty$. Existence of $q$ and $\eta$ therefore contradicts existence of $\widetilde{x}$. However, the result holds even if $(DLP)$ is infeasible.)

(d) If $\widetilde{y}$, $\widetilde{s}$ is feasible to $(DLP)$, it is potentially an optimal solution. There is no contradiction to optimality by $\widetilde{x}^T\widetilde{s} = 1$. It is possible that $\widetilde{y}$, $\widetilde{s}$ is optimal to $(DLP)$, in which case $\widetilde{x}$ is not optimal to $(PLP)$.

**2.** (a) The optimality conditions of $(P_\mu)$ may be written as

$$\begin{aligned} c - \mu X^{-1}e &= A^T\lambda, \\ Ax &= b, \end{aligned}$$

in addition to $x > 0$, where $X = \text{diag}(x)$ and $e$ is the vector of ones. By letting $y = \lambda$ and $s = \mu X^{-1}e$, these equations are equivalent to the primal-dual nonlinear equations

$$\begin{aligned} A^Ty + s &= c, \\ Ax &= b, \\ XSe &= \mu e, \end{aligned}$$

with $S = \text{diag}(s)$, since $s = \mu X^{-1}e$ if and only if $XSe = \mu e$. Therefore, $x(\mu) = \widetilde{x}$, $y(\mu) = \tilde{\lambda}$ and $s(\mu) = \mu\tilde{X}^{-1}e$. By the given numbers, we obtain

$$x(0.01) \approx \begin{pmatrix} 3.0136 \\ 1.9828 \\ 0.0085 \\ 0.0046 \\ 0.0120 \\ 0.9812 \end{pmatrix} \quad \text{and} \quad y(0.01) \approx \begin{pmatrix} 0.9898 \\ -0.9505 \\ 0.9437 \end{pmatrix}.$$

It is not straightforward to calculate $s(0.01)$ without a calculator. We could calculate $s(0.01)$ either as $\mu\tilde{X}^{-1}e$ or as $c - A^T\tilde{\lambda}$. This is not required.

If this would be done, the result is

$$s(0.01) \approx \begin{pmatrix} 0.0033 \\ 0.0050 \\ 1.1774 \\ 2.1588 \\ 0.8327 \\ 0.0102 \end{pmatrix}.$$

(b) We expect the solution of the primal-dual nonlinear equations $x(\mu)$, $y(\mu)$, $s(\mu)$, to differ by the order of $\mu$ from a primal-dual optimal pair $x^*$, $y^*$ and $s^*$ to $(PLP)$ and $(DLP)$ respectively. By rounding $x(0.01)$ and $y(0.01)$ to nearest integer, we obtain

$$x^* = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad y^* = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Given $y^*$, we may then calculate $s^*$ as

$$s^* = c - A^T y^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

By verifying $Ax^* = b$, $x^* \geq 0$, $s^* \geq 0$ and $x^{*T}s^* = 0$, it follows that $x^*$ is optimal to $(PLP)$ and $y^*$, $s^*$ is optimal to $(DLP)$.

(c) Yes, because the solution $x^*$ is a primal nondegenerate basic feasible solution and $y^*$, $s^*$ is a dual nondegenerate basic feasible solution. To see this, consider the submatrix of $A$ corresonding to positive components in $x^*$,

$$\begin{pmatrix} A_1 & A_2 & A_6 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 \\ -1 & 3 & 0 \\ -1 & 2 & 0 \end{pmatrix}.$$

This is a square and nonsingular matrix, since its determinant is nonzero.

**3.** (See the course material.)

**4.** The suggested initial extreme points $v_1 = (1 \;\; 1 \; {-1} \; {-1})^T$ and $v_2 = (1 \;\; 1 \;\; 1 \;\; 1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} Av_1 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (2 \; 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{5}{8} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1 \; c^T v_2) = (-8 \;\; -2)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} 7 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -8 \\ -2 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ -\frac{11}{4} \end{pmatrix}.$$

By forming $c^T - y_1 A = (-3/2 \;\; -5/4 \;\; -1/2 \;\; 1/2)$ we obtain the subproblem

$$\begin{array}{lll} \frac{11}{4}+ & \text{minimize} & -\frac{3}{2}x_1 - \frac{5}{4}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ & \text{subject to} & -1 \le x_j \le 1, \quad j = 1, \dots, 4. \end{array}$$

An optimal extreme point to the subproblem is given by $v_3 = (1 \;\; 1 \;\; 1 \;\; -1)^T$ with optimal value -1. Hence, $\alpha_3$ should enter the basis. The corresponding column in the master problem is given by

$$\begin{pmatrix} Av_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The change to the basic variables is given by

$$\begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = -\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

Finding the maximum step $\eta$ for which $\alpha + \eta p \ge 0$ gives

$$\begin{pmatrix} \frac{3}{8} \\ \frac{5}{8} \end{pmatrix} + \eta \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., $\eta = 3/4$ so that $\alpha_1$ leaves the basis.

Hence, the new basis corresponds to $v_2$ and $v_3$ so that

$$B = \begin{pmatrix} Av_2 & Av_3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}.$$

The basic variables are given by

$$\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_2 \; c^T v_3) = (-2 \;\; -6)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

By forming $c^T - y_1 A = (-1 \;\; -1 \;\; -1 \quad 0)$ we obtain the subproblem

$$\begin{array}{lll} 3+ & \text{minimize} & -x_1 - x_2 - x_3 \\ & \text{subject to} & -1 \le x_j \le 1, \quad j = 1, \dots, 4. \end{array}$$

Both $v_2$ and $v_3$ are optimal extreme points to the subproblem, so the optimal value of the subproblem is 0. Hence, the master problem has been solved. The solution to the original problem is given by

$$v_2\alpha_2 + v_3\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{4} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \frac{3}{4} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix}.$$

The optimal value is –5.

**5.** *Remark:* An application of Lagrangian decomposition can be found in M. Prytz and A. Forsgren, *Dimensioning multicast-enabled communications networks*, Networks 39 (2002), 216–231.

(a) We have $\text{optval}(D_1) = \underset{u \ge 0}{\text{maximize}}\ \varphi_1(u)$. We may now follow the steps in the hint and explain each equality.
By the definition of $(D_1)$, we obtain

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_1(u) = \underset{u\ge 0}{\text{maximize}} \left\{ \begin{array}{ll} \text{minimize} & c^T x - u^T(Ax - b) \\ \text{subject to} & Cx \ge d, x \ge 0, \ x \text{ integer} \end{array} \right\}.$$

By noting that $b^T u$ is independent of $x$, we may rewrite as

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_1(u) = \underset{u\ge 0}{\text{maximize}} \left\{ \begin{array}{lll} b^T u + & \text{minimize} & (c - A^T u)^T x \\ & \text{subject to} & Cx \ge d, x \ge 0, x \text{ integer} \end{array} \right\}.$$

Since the extreme points of $\text{conv}\{x : Cx \ge d, x \ge 0, \ x \text{ integer}\}$ belong to $\{x : Cx \ge d, x \ge 0, \ x \text{ integer}\}$, and the minimizer of a linear function can be found at an extreme point, we may equivalently write

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_1(u) = \underset{u\ge 0}{\text{maximize}} \left\{ \begin{array}{lll} b^T u + & \text{minimize} & (c - A^T u)^T x \\ & \text{subject to} & x \in \text{conv}\{Cx \ge d, x \ge 0, x \text{ integer}\} \end{array} \right\}.$$

Making use of the identity

$$\text{conv}\{x : Cx \ge d, x \ge 0, x \text{ integer}\} = \{x : \bar{C}x \ge \bar{d}\},$$

gives

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_1(u) = \underset{u\ge 0}{\text{maximize}} \left\{ \begin{array}{lll} b^Tu + & \text{minimize} & (c - A^Tu)^Tx \\ & \text{subject to} & \bar{C}x \ge \bar{d}. \end{array} \right\}.$$

By strong duality, the primal-dual pairs of linear programs

$$\begin{array}{ll} \text{minimize} & (c - A^Tu)^Tx \\ \text{subject to} & \bar{C}x \ge \bar{d}, \end{array} \quad \text{and} \quad \begin{array}{ll} \text{maximize} & \bar{d}^T\bar{u} \\ \text{subject to} & \bar{C}^T\bar{u} = c - A^Tu,\ \bar{u} \ge 0, \end{array}$$

have the same optimal value. Therefore,

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_1(u) = \underset{u\ge 0}{\text{maximize}} \left\{ \begin{array}{lll} b^Tu + & \text{maximize} & \bar{d}^T\bar{u} \\ & \text{subject to} & \bar{C}^T\bar{u} = c - A^Tu,\ \bar{u} \ge 0, \end{array} \right\}$$

The inner and outer maximizations may now be written as one maximization, so that

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_1(u) = \begin{array}{ll} \text{maximize} & b^Tu + \bar{d}^T\bar{u} \\ \text{subject to} & A^Tu + \bar{C}^T\bar{u} = c,\ u \ge 0,\ \bar{u} \ge 0, \end{array}$$

Again, by strong duality, the primal-dual pairs of linear programs

$$\begin{array}{ll} \text{maximize} & b^Tu + \bar{d}^T\bar{u} \\ \text{subject to} & A^Tu + \bar{C}^T\bar{u} = c, \\ & u \ge 0,\ \bar{u} \ge 0, \end{array} \quad \text{and} \quad \begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & Ax \ge b, \\ & \bar{C}x \ge \bar{d}, \end{array}$$

have the same optimal value. Therefore,

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_1(u) = \begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & Ax \ge b, \\ & \bar{C}x \ge \bar{d}, \end{array}$$

Again, using the equivalence

$$\text{conv}\{x : Cx \ge d, x \ge 0, x \text{ integer}\} = \{x : \bar{C}x \ge \bar{d}\},$$

we obtain

$$\underset{u\ge 0}{\text{maximize}}\ \varphi_3(u) = \begin{array}{ll} \text{minimize} & c^Tx \\ & Ax \ge b, \\ & x \in \text{conv}\{x : Cx \ge d, x \ge 0,\ x \text{ integer}\}, \end{array}$$

which is the required result.

The result on optval$(D_2)$ follows by replacing the roles of $A$ and $b$ by $C$ and $d$.

(b) We may carry out the analogous analysis on $(D_3)$. Analogous to $\bar{C}$ and $\bar{d}$, let $\bar{A}$ and $\bar{b}$ be a matrix and vector such that

$$\text{conv}\{x : Ax \ge b, x \ge 0, x \text{ integer}\} = \{x : \bar{A}x \ge \bar{b}\}.$$

By the definition of $(D_3)$, we obtain

$$\underset{w}{\text{maximize}}\ \varphi_3(w) = \underset{w}{\text{maximize}} \left\{ \begin{array}{ll} \text{minimize} & c^Tx - w^T(x - y) \\ \text{subject to} & x \in \text{conv}\{x : Ax \ge b, x \ge 0,\ x \text{ integer}\}, \\ & y \in \text{conv}\{x : Cx \ge d, x \ge 0,\ x \text{ integer}\}. \end{array} \right\}.$$

By using the equivalences

$$\begin{aligned}\text{conv}\{x : Ax \geq b, x \geq 0, x \text{ integer}\} &= \{x : \bar{A}x \geq \bar{b}\},\\ \text{conv}\{x : Cx \geq d, x \geq 0, x \text{ integer}\} &= \{x : \bar{C}x \geq \bar{d}\},\end{aligned}$$

it follows that

$$\underset{w}{\text{maximize}}\ \varphi_3(w) = \underset{w}{\text{maximize}} \left\{ \begin{array}{lllll} \text{minimize} & (c-w)^T x & + & \text{minimize} & w^T y \\ \text{subject to} & \bar{A}x \geq \bar{b}. & & \text{subject to} & \bar{C}x \geq \bar{d}. \end{array} \right\}.$$

By strong duality, the primal-dual pairs of linear programs

$$\begin{array}{ll} \text{minimize} & (c-w)^T x \\ \text{subject to} & \bar{A}x \geq \bar{b}, \end{array} \quad \text{and} \quad \begin{array}{ll} \text{maximize} & \bar{b}^T \bar{v} \\ \text{subject to} & \bar{A}^T \bar{v} = c - w,\ \bar{v} \geq 0, \end{array}$$

have the same optimal value. The same is true for

$$\begin{array}{ll} \text{minimize} & w^T y \\ \text{subject to} & \bar{C}x \geq \bar{d}, \end{array} \quad \text{and} \quad \begin{array}{ll} \text{maximize} & \bar{d}^T \bar{u} \\ \text{subject to} & \bar{C}^T \bar{u} = w,\ \bar{u} \geq 0. \end{array}$$

Therefore,

$$\underset{w}{\text{maximize}}\ \varphi_3(w) = \underset{w}{\text{maximize}} \left\{ \begin{array}{lllll} \text{maximize} & \bar{b}^T \bar{v} & + & \text{maximize} & \bar{d}^T \bar{u} \\ \text{subject to} & \bar{A}^T \bar{v} = c - w, & & \text{subject to} & \bar{C}^T \bar{u} = w, \\ & \bar{v} \geq 0. & & & \bar{u} \geq 0. \end{array} \right\}$$

The inner and outer maximizations may now be written as one maximization, so that

$$\underset{w}{\text{maximize}}\ \varphi_3(w) = \begin{array}{ll} \text{maximize} & \bar{b}^T \bar{v} + \bar{d}^T \bar{u} \\ \text{subject to} & \bar{A}^T \bar{v} + w = c, \\ & \bar{C}^T \bar{u} - w = 0, \\ & \bar{v} \geq 0,\ \bar{u} \geq 0. \end{array}$$

We may eliminate $w$ from the optimization problem by $w = \bar{C}^T \bar{u}$, so that

$$\underset{w}{\text{maximize}}\ \varphi_3(w) = \begin{array}{ll} \text{maximize} & \bar{b}^T \bar{v} + \bar{d}^T \bar{u} \\ \text{subject to} & \bar{A}^T \bar{v} + \bar{C}^T \bar{u} = c, \\ & \bar{v} \geq 0,\ \bar{u} \geq 0. \end{array}$$

Again, by strong duality, the primal-dual pairs of linear programs

$$\begin{array}{ll} \text{maximize} & \bar{b}^T \bar{v} + \bar{d}^T \bar{u} \\ \text{subject to} & \bar{A}^T \bar{v} + \bar{C}^T \bar{u} = c, \\ & \bar{v} \geq 0,\ \bar{u} \geq 0, \end{array} \quad \text{and} \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x \geq \bar{b}, \\ & \bar{C}x \geq \bar{d}, \end{array}$$

have the same optimal value. Therefore,

$$\underset{w}{\text{maximize}}\ \varphi_3(w) = \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x \geq \bar{b}, \\ & \bar{C}x \geq \bar{d}. \end{array}$$

Again, using the equivalences

$$\text{conv}\{x : Ax \geq b, x \geq 0, x \text{ integer}\} = \{x : \bar{A}x \geq \bar{b}\},$$
$$\text{conv}\{x : Cx \geq d, x \geq 0, x \text{ integer}\} = \{x : \bar{C}x \geq \bar{d}\},$$

we obtain

$$\begin{array}{llll} \underset{w}{\text{maximize}}\ \varphi_3(w) = & \text{minimize} & c^T x \\ & \text{subject to} & x \in \text{conv}\{x : Ax \geq b, x \geq 0,\ x \text{ integer}\}, \\ & & x \in \text{conv}\{x : Cx \geq d, x \geq 0,\ x \text{ integer}\}, \end{array}$$

which is the required result.
A shorter proof would be to say that we conclude from Question 5a that the optimal value of the Lagrangian dual problem may be obtained as the minimizer of the linear function subject to the relaxed constraints plus the convex hull of the unrelaxed constraints, i.e.,

$$\begin{array}{llll} \underset{w}{\text{maximize}}\ \varphi_3(w) = & \text{minimize} & c^T x \\ & \text{subject to} & x = y, \\ & & x \in \text{conv}\{x : Ax \geq b, x \geq 0,\ x \text{ integer}\}, \\ & & y \in \text{conv}\{x : Cx \geq d, x \geq 0,\ x \text{ integer}\}. \end{array}$$

Since $x = y$, we may eliminate $y$ so that

$$\begin{array}{llll} \underset{w}{\text{maximize}}\ \varphi_3(w) = & \text{minimize} & c^T x \\ & \text{subject to} & x \in \text{conv}\{x : Ax \geq b, x \geq 0,\ x \text{ integer}\}, \\ & & x \in \text{conv}\{x : Cx \geq d, x \geq 0,\ x \text{ integer}\}, \end{array}$$

as required.
Since

$$\text{conv}\{x : Ax \geq b, x \geq 0, x \text{ integer}\} \subseteq \{x : Ax \geq b\},$$

we conclude that optval$(D_1)$ is the optimal value of a linear program which is a relaxation of a linear program that gives the optimal value of $(D_3)$. Therefore, optval$(D_3) \geq$ optval$(D_1)$. By a symmetric argument, optval$(D_3) \geq$ optval$(D_2)$.