Interactive Theorem Proving (ITP) Course

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Part I

Introduction

Motivation

- Complex systems almost certainly contain bugs.
- Critical systems (e. g. avionics) need to meet very high standards.
- It is infeasible in practice to achieve such high standards just by testing.
- Debugging via testing suffers from diminishing returns.

> **“Program testing can be used to show the presence of bugs, but never to show their absence!”**
> **— Edsger W. Dijkstra**

Famous Bugs

- Pentium FDIV bug (1994)
  (missing entry in lookup table, \$475 million damage)
- Ariane V explosion (1996)
  (integer overflow, \$1 billion prototype destroyed)
- Mars Climate Orbiter (1999)
  (destroyed in Mars orbit, mixup of units pound-force and newtons)
- Knight Capital Group Error in Ultra Short Time Trading (2012)
  (faulty deployment, repurposing of critical flag, \$440 lost in 45 min on stock exchange)
- ...

Fun to read

```
http://www.cs.tau.ac.il/~nachumd/verify/horror.html
https://en.wikipedia.org/wiki/List_of_software_bugs
```

Proof

- proof can show absence of errors in design
- but proofs talk about a **design**, not a **real system**
- ⇒ testing and proving complement each other

> **“As far as the laws of mathematics**
> **refer to reality, they are not certain;**
> **and as far as they are certain,**
> **they do not refer to reality.”**
> **— Albert Einstein**

Mathematical vs. Formal Proof

Mathematical Proof

- informal, convince other mathematicians
- checked by community of domain experts
- subtle errors are hard to find
- often provide some new insight about our world
- often short, but require creativity and a brilliant idea

Formal Proof

- formal, rigorously use a logical formalism
- checkable by *stupid* machines
- very reliable
- often contain no new ideas and no amazing insights
- often long, very tedious, but largely trivial

We are interested in formal proofs in this lecture.

Detail Level of Formal Proof

In **Principia Mathematica** it takes 300 pages to prove 1+1=2.

This is nicely illustrated in **Logicomix - An Epic Search for Truth**.

WHITEHEAD, I GOT IT THIS TIME!
I'M FINALLY DONE WITH THE DAMN THING!
ARE YOU, REALLY?
YES, IT'S PROVEN...
1+1=2!
?

Automated vs Manual (Formal) Proof

Fully Manual Proof

- very tedious one has to grind through many trivial but detailed proofs
- easy to make mistakes
- hard to keep track of all assumptions and preconditions
- hard to maintain, if something changes (see Ariane V)

Automated Proof

- amazing success in certain areas
- but still often infeasible for interesting problems
- hard to get insights in case a proof attempt fails
- even if it works, it is often not that automated
  - run automated tool for a few days
  - abort, change command line arguments to use different heuristics
  - run again and iterate till you find a set of heuristics that prove it fully automatically in a few seconds

Interactive Proofs

- combine strengths of manual and automated proofs
- many different options to combine automated and manual proofs
  - mainly check existing proofs (e. g. HOL Zero)
  - user mainly provides lemmata statements, computer searches proofs using previous lemmata and very few hints (e. g. ACL 2)
  - most systems are somewhere in the middle
- typically the human user
  - provides insights into the problem
  - structures the proof
  - provides main arguments
- typically the computer
  - checks proof
  - keeps track of all use assumptions
  - provides automation to grind through lengthy, but trivial proofs

Typical Interactive Proof Activities

- provide precise definitions of concepts
- state properties of these concepts
- prove these properties
  - human provides insight and structure
  - computer does book-keeping and automates simple proofs
- build and use libraries of formal definitions and proofs
  - formalisations of mathematical theories like
    - lists, sets, bags, . . .
    - real numbers
    - probability theory
  - specifications of real-world artefacts like
    - processors
    - programming languages
    - network protocols
  - reasoning tools

There is a strong connection with programming.
Lessons learned in Software Engineering apply.

Different Interactive Provers

- there are many different interactive provers, e. g.
  - Isabelle/HOL
  - Coq
  - PVS
  - HOL family of provers
  - ACL2
  - ...
- important differences
  - the formalism used
  - level of trustworthiness
  - level of automation
  - libraries
  - languages for writing proofs
  - user interface
  - ...

Which theorem prover is the best one? :-)

- there is no best theorem prover
- better question: Which is the best one for a certain purpose?
- important points to consider
  - existing libraries
  - used logic
  - level of automation
  - user interface
  - importance development speed versus trustworthiness
  - How familiar are you with the different provers?
  - Which prover do people in your vicinity use?
  - your personal preferences
  - ...

In this course we use the HOL theorem prover, because it is used by the TCS group.

Part II

Organisational Matters

Aims of this Course

Aims

- introduction to interactive theorem proving (ITP)
- being able to evaluate whether a problem can benefit from ITP
- hands-on experience with HOL
- learn how to build a formal model
- learn how to express and prove important properties of such a model
- learn about basic conformance testing
- use a theorem prover on a small project

Required Prerequisites

- some experience with functional programming
- knowing Standard ML syntax
- basic knowledge about logic (e. g. First Order Logic)

Dates

- Interactive Theorem Proving Course takes place in Period 4 of the academic year 2016/2017
- always in room 4523 or 4532
- each week

Mondays	10:15 - 11:45	lecture
Wednesdays	10:00 - 12:00	practical session
Fridays	13:00 - 15:00	practical session

- no lecture on Monday, 1st of May, instead on Wednesday, 3rd May
- last lecture: 12th of June
- last practical session: 21st of June
- 9 lectures, 17 practical sessions

Exercises

- after each lecture an exercise sheet is handed out
- work on these exercises alone, except if stated otherwise explicitly
- exercise sheet contains due date
  - usually 10 days time to work on it
  - hand in during practical sessions
  - lecture Monday $\longrightarrow$ hand in at latest in next week's Friday session
- main purpose: understanding ITP and learn how to use HOL
  - no detailed grading, just pass/fail
  - retries possible till pass
  - if stuck, ask me or one another
  - practical sessions intend to provide this opportunity

Practical Sessions

- very informal
- main purpose: work on exercises
  - I have a look and provide feedback
  - you can ask questions
  - I might sometimes explain things not covered in the lectures
  - I might provide some concrete tips and tricks
  - you can also discuss with each other
- attendance not required, but highly recommended
  - exception: session on 21st April
- only requirement: turn up long enough to hand in exercises
- you need to bring your own computer

Handing-in Exercises

- exercises are intended to be handed-in during practical sessions
- attend at least one practical session each week
- leave reasonable time to discuss exercises
    - don't try to hand your solution in Friday 14:55
- retries possible, but reasonable attempt before deadline required
- handing-in outside practical sessions
    - only if you have a good reason
    - decided on a case-by-case basis
- electronic hand-ins
    - only to get detailed feedback
    - does not replace personal hand-in
    - exceptions on a case-by-case basis if there is a good reason
    - I recommend using a KTH GitHub repo

Passing the ITP Course

- there is only a pass/fail mark
- to pass you need to
  - attend at least 7 of the 9 lectures
  - pass 8 of the 9 exercises

Communication

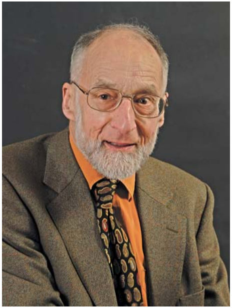
- we have the advantage of being a small group
- therefore we are flexible
- so please ask questions, even during lectures
- there are many shy people, therefore
  - anonymous checklist after each lecture
  - anonymous background questionnaire in first practical session
- further information is posted on **Interactive Theorem Proving Course** group on Group Web
- contact me (Thomas Tuerk) directly, e. g. via email `thomas@kth.se`

Part III

HOL 4 History and Architecture

LCF - Logic of Computable Functions

- **Standford LCF** 1971-72 by Milner et al.
- formalism devised by Dana Scott in 1969
- intended to reason about recursively defined functions
- intended for computer science applications
- strengths
  - powerful simplification mechanism
  - support for backward proof
- limitations
  - proofs need a lot of memory
  - fixed, hard-coded set of proof commands



Robin Milner
(1934 - 2010)

LCF - Logic of Computable Functions II

- Milner worked on improving LCF in Edinburgh
- research assistants
  - Lockwood Morris
  - Malcolm Newey
  - Chris Wadsworth
  - Mike Gordon
- **Edinburgh LCF** 1979
- introduction of **Meta Language** (ML)
- ML was invented to write proof procedures
- ML become an influential functional programming language
- using ML allowed implementing the **LCF approach**

LCF Approach

- implement an abstract datatype thm to represent theorems
- semantics of ML ensure that values of type thm can only be created using its interface
- interface is very small
  - predefined theorems are axioms
  - function with result type theorem are inferences
- $\Longrightarrow$ However you create a theorem, it is valid.
- together with similar abstract datatypes for types and terms, this forms the kernel

LCF Approach II

Modus Ponens Example

Inference Rule

$$\frac{\Gamma \vdash a \Rightarrow b \qquad \Delta \vdash a}{\Gamma \cup \Delta \vdash b}$$

SML function

```
val MP : thm -> thm -> thm
```

$\mathtt{MP}(\Gamma \vdash a \Rightarrow b)(\Delta \vdash a) = (\Gamma \cup \Delta \vdash b)$

- very trustworthy — only the small kernel needs to be trusted
- efficient — no need to store proofs

Easy to extend and automate

However complicated and potentially buggy your code is, if a value of type theorem is produced, it has been created through the small trusted interface. Therefore the statement really holds.

LCF Style Systems

There are now many interactive theorem provers out there that use an approach similar to that of Edinburgh LCF.

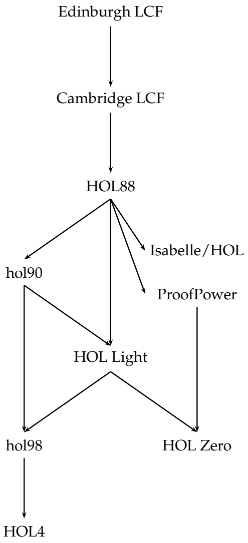
- HOL family
  - HOL theorem prover
  - HOL Light
  - HOL Zero
  - Proof Power
  - ...
- Isabelle
- Nuprl
- Coq
- ...

History of HOL

- 1979 Edinburgh LCF by Milner, Gordon, et al.
- 1981 Mike Gordon becomes lecturer in Cambridge
- 1985 Cambridge LCF
  - Larry Paulson and Gèrard Huet
  - implementation of ML compiler
  - powerful simplifier
  - various improvements and extensions
- 1988 HOL
  - Mike Gordon and Keith Hanna
  - adaption of Cambridge LCF to classical higher order logic
  - intention: hardware verification
- 1990 HOL90
  reimplementation in SML by Konrad Slind at University of Calgary
- 1998 HOL98
  implementation in Moscow ML and new library and theory mechanism
- since then HOL Kananaskis releases, called informally HOL 4

Family of HOL

- **ProofPower**
  commercial version of HOL88 by Roger Jones, Rob Arthan et al.
- **HOL Light**
  lean CAML / OCaml port by John Harrison
- **HOL Zero**
  trustworthy proof checker by Mark Adams
- **Isabelle**
  - 1990 by Larry Paulson
  - meta-theorem prover that supports multiple logics
  - however, mainly HOL used, ZF a little
  - nowadays probably the most widely used HOL system
  - originally designed for software verification



Part IV

HOL's Logic

HOL Logic

- the HOL theorem prover uses a version of classical **h**igher **o**rder **l**ogic: classical higher order predicate calculus with terms from the typed lambda calculus (i. e. simple type theory)
- this sounds complicated, but is intuitive for SML programmers
- (S)ML and HOL logic designed to fit each other
- if you understand SML, you understand HOL logic

HOL = functional programming + logic

Ambiguity Warning

The acronym *HOL* refers to both the *HOL interactive theorem prover* and the *HOL logic* used by it. It's also a common abbreviation for *higher order logic* in general.

Types

- SML datatype for types
  - **Type Variables** (`'a`, α, `'b`, β, …)
    Type variables are implicitly universally quantified. Theorems containing type variables hold for all instantiations of these. Proofs using type variables can be seen as proof schemata.
  - **Atomic Types** (`c`)
    Atomic types denote fixed types. Examples: `num`, `bool`, `unit`
  - **Compound Types** ($(\sigma_1, \ldots, \sigma_n) op$)
    op is a type operator of arity n and $\sigma_1, \ldots, \sigma_n$ argument types. Type operators denote operations for constructing types.
    Examples: `num list` or `'a # 'b`.
  - **Function Types** ($\sigma_1 \to \sigma_2$)
    $\sigma_1 \to \sigma_2$ is the type of total functions from σ_1 to σ_2.
- types are never empty in HOL, i. e.
  for each type at least one value exists
- all HOL functions are total

Terms

- SML datatype for terms
  - **Variables** ($\mathtt{x}, \mathtt{y}, \ldots$)
  - **Constants** ($\mathtt{c}, \ldots$)
  - **Function Application** (`f a`)
  - **Lambda Abstraction** (`\x. f x` or $\lambda x.\ fx$)
    Lambda abstraction represents anonymous function definition.
    The corresponding SML syntax is `fn x => f x`.
- terms have to be well-typed
- same typing rules and same type-inference as in SML take place
- terms very similar to SML expressions
- notice: predicates are functions with return type `bool`, i. e. no distinction between functions and predicates, terms and formulae

Terms II

HOL term	SML expression	type HOL / SML
0	0	num / int
x:'a	x:'a	variable of type 'a
x:bool	x:bool	variable of type bool
x + 5	x + 5	applying function + to x and 5
\x. x + 5	fn x => x + 5	anonymous (a. k. a. inline) function of type num -> num
(5, T)	(5, true)	num # bool / int * bool
[5;3;2]++[6]	[5,3,2]@[6]	num list / int list

Free and Bound Variables / Alpha Equivalence

- in SML, the names of function arguments does not matter (much)
- similarly in HOL, the names of variables used by lambda-abstractions does not matter (much)
- the lambda-expression $\lambda x.\ t$ is said to **bind** the variables x in term t
- variables that are guarded by a lambda expression are called **bound**
- all other variables are **free**
- Example: x is free and y is bound in $(x = 5) \wedge (\lambda y.\ (y < x))$ 3
- the names of bound variables are unimportant semantically
- two terms are called **alpha-equivalent** iff they differ only in the names of bound variables
- Example: $\lambda x.\ x$ and $\lambda y.\ y$ are alpha-equivalent
- Example: x and y are not alpha-equivalent

Theorems

- theorems are of the form $\Gamma \vdash p$ where
  - Γ is a set of hypothesis
  - p is the conclusion of the theorem
  - all elements of Γ and p are formulae, i. e. terms of type `bool`
- $\Gamma \vdash p$ records that using Γ the statement p has been proved
- notice difference to logic: there it means can be proved
- the proof itself is not recorded
- theorems can only be created through a small interface in the **kernel**

HOL Light Kernel

- the HOL kernel is hard to explain
  - for historic reasons some concepts are represented rather complicated
  - for speed reasons some derivable concepts have been added
- instead consider the HOL Light kernel, which is a cleaned-up version
- there are two predefined constants
  - `= : 'a -> 'a -> bool`
  - `@ : ('a -> bool) -> 'a`
- there are two predefined types
  - `bool`
  - `ind`
- the meaning of these types and constants is given by inference rules and axioms

HOL Light Inferences I

$$\frac{}{\vdash t = t}\ \text{REFL}$$

$$\frac{\begin{array}{c}\Gamma \vdash s = t \\ \Delta \vdash t = u\end{array}}{\Gamma \cup \Delta \vdash s = u}\ \text{TRANS}$$

$$\frac{\begin{array}{c}\Gamma \vdash s = t \\ \Delta \vdash u = v \\ \textit{types fit}\end{array}}{\Gamma \cup \Delta \vdash s(u) = t(v)}\ \text{COMB}$$

$$\frac{\begin{array}{c}\Gamma \vdash s = t \\ x \ \textit{not free in}\ \Gamma\end{array}}{\Gamma \vdash \lambda x.\ s = \lambda x.\ t}\ \text{ABS}$$

$$\frac{}{\vdash (\lambda x.\ t)x = t}\ \text{BETA}$$

$$\frac{}{\{p\} \vdash p}\ \text{ASSUME}$$

HOL Light Inferences II

$$\frac{\Gamma \vdash p \Leftrightarrow q \qquad \Delta \vdash p}{\Gamma \cup \Delta \vdash q} \text{ EQ\_MP}$$

$$\frac{\Gamma \vdash p \qquad \Delta \vdash q}{(\Gamma - \{q\}) \cup (\Delta - \{p\}) \vdash p \Leftrightarrow q} \text{ DEDUCT\_ANTISYM\_RULE}$$

$$\frac{\Gamma[x_1, \ldots, x_n] \vdash p[x_1, \ldots, x_n]}{\Gamma[t_1, \ldots, t_n] \vdash p[t_1, \ldots, t_n]} \text{ INST}$$

$$\frac{\Gamma[\alpha_1, \ldots, \alpha_n] \vdash p[\alpha_1, \ldots, \alpha_n]}{\Gamma[\gamma_1, \ldots, \gamma_n] \vdash p[\gamma_1, \ldots, \gamma_n]} \text{ INST\_TYPE}$$

HOL Light Axioms and Definition Principles

- 3 axioms needed

ETA_AX	$\mid-(\lambda x.\ t\ x) = t$
SELECT_AX	$\mid-P\ x \Longrightarrow P((@)P))$
INFINITY_AX	predefined type `ind` is infinite

- definition principle for constants
  - constants can be introduced as abbreviations
  - constraint: no free vars and no new type vars
- definition principle for types
  - new types can be defined as non-empty subtypes of existing types
- both principles
  - lead to conservative extensions
  - preserve consistency

HOL Light derived concepts

Everything else is derived from this small kernel.

$$
\begin{array}{ccl}
T & =_{def} & (\lambda p.\ p) = (\lambda p.\ p) \\
\wedge & =_{def} & \lambda p\, q.\ (\lambda f.\ f\ \ p\ \ q) = (\lambda f.\ f\ \ T\ \ T) \\
\Longrightarrow & =_{def} & \lambda p\, q.\ (p \wedge q \Leftrightarrow p) \\
\forall & =_{def} & \lambda P.\ (P = \lambda x.\ T) \\
\exists & =_{def} & \lambda P.\ (\forall q.\ (\forall x.\ P(x) \Longrightarrow q) \Longrightarrow q) \\
\ldots & &
\end{array}
$$

Multiple Kernels

- Kernel defines abstract datatypes for types, terms and theorems
- one does not need to look at the internal implementation
- therefore, easy to exchange
- there are at least 3 different kernels for HOL
  - standard kernel (de Bruijn indices)
  - experimental kernel (name / type pairs)
  - OpenTheory kernel (for proof recording)

HOL Logic Summary

- HOL theorem prover uses classical higher order logic
- HOL logic is very similar to SML
  - syntax
  - type system
  - type inference
- HOL theorem prover very trustworthy because of LCF approach
  - there is a small kernel
  - proofs are not stored explicitly
- you don't need to know the details of the kernel
- usually one works at a much higher level of abstraction

Part V

Basic HOL Usage

HOL Technical Usage Issues

- practical issues are discussed in practical sessions
  - how to install HOL
  - which key-combinations to use in emacs-mode
  - detailed signature of libraries and theories
  - all parameters and options of certain tools
  - ...
- exercise sheets sometimes
  - ask to read some documentation
  - provide examples
  - list references where to get additional information
- if you have problems, ask me outside lecture (tuerk@kth.se)
- covered only very briefly in lectures

Installing HOL

- webpage: `https://hol-theorem-prover.org`
- HOL supports two SML implementations
  - Moscow ML (`http://mosml.org`)
  - PolyML (`http://www.polyml.org`)
- I recommend using PolyML
- please use emacs with
  - hol-mode
  - sml-mode
  - hol-unicode, if you want to type Unicode
- please install recent revision from git repo or Kananaskis 11 release
- documentation found on HOL webpage and with sources

General Architecture

- HOL is a collection of SML modules
- starting HOL starts a SML Read-Eval-Print-Loop (REPL) with
  - some HOL modules loaded
  - some default modules opened
  - an input wrapper to help parsing terms called `unquote`
- `unquote` provides special quotes for terms and types
  - implemented as input filter
  - `‘‘my-term‘‘` becomes `Parse.Term [QUOTE "my-term"]`
  - `‘‘:my-type‘‘` becomes `Parse.Type [QUOTE ":my-type"]`
- main interfaces
  - **emacs** (used in the course)
  - vim
  - bare shell

Filenames

- `*Script.sml` — HOL proof script file
  - script files contain definitions and proof scripts
  - executing them results in HOL searching and checking proofs
  - this might take very long
  - resulting theorems are stored in `*Theory.{sml|sig}` files
- `*Theory.{sml|sig}` — HOL theory
  - auto-generated by corresponding script file
  - load quickly, because they don't search/check proofs
  - do not edit theory files
- `*Syntax.{sml|sig}` — syntax libraries
  - contain syntax related functions
  - i. e. functions to construct and destruct terms and types
- `*Lib.{sml|sig}` — general libraries
- `*Simps.{sml|sig}` — simplifications
- `selftest.sml` — selftest for current directory

Directory Structure

- `bin` — HOL binaries
- `src` — HOL sources
- `examples` — HOL examples
  - interesting projects by various people
  - examples owned by their developer
  - coding style and level of maintenance differ a lot
- `help` — sources for reference manual
  - after compilation home of reference HTML page
- `Manual` — HOL manuals
  - Tutorial
  - Description
  - Reference (PDF version)
  - Interaction
  - Quick (cheat pages)
  - Style-guide
  - ...

Unicode

- HOL supports both Unicode and pure ASCII input and output
- advantages of Unicode compared to ASCII
  - easier to read (good fonts provided)
  - no need to learn special ASCII syntax
- disadvanges of Unicode compared to ASCII
  - harder to type (even with `hol-unicode.el`)
  - less portable between systems
- whether you like Unicode is highly a matter of personal taste
- HOL's policy
  - no Unicode in HOL's source directory `src`
  - Unicode in examples directory `examples` is fine
- I recommend turning Unicode output off initially
  - this simplifies learning the ASCII syntax
  - no need for special fonts
  - it is easier to copy and paste terms from HOL's output

Where to find help?

- reference manual
  - available as HTML pages, single PDF file and in-system help
- description manual
- Style-guide (still under development)
- HOL webpage (`https://hol-theorem-prover.org`)
- mailing-list `hol-info`
- `DB.match` and `DB.find`
- `*Theory.sig` and `selftest.sml` files
- ask someone, e. g. me :-) (tuerk@kth.se)

Part VI

Forward Proofs

Kernel too detailed

- we already discussed the HOL Logic
- the kernel itself does not even contain basic logic operators
- usually one uses a much higher level of abstraction
  - many operations and datatypes are defined
  - high-level derived inference rules are used
- let’s now look at this more common abstraction level

Common Terms and Types

	Unicode	**ASCII**
type vars	α, β, . . .	`'a, 'b, ...`
type annotated term	`term:type`	`term:type`
true	`T`	`T`
false	`F`	`F`
negation	`¬b`	`~b`
conjunction	`b1 ∧ b2`	`b1 /\ b2`
disjunction	`b1 ∨ b2`	`b1 \/ b2`
implication	`b1 ⟹ b2`	`b1 ==> b2`
equivalence	`b1 ⟺ b2`	`b1 <=> b2`
disequation	`v1 ≠ v2`	`v1 <> v2`
all-quantification	`∀x. P x`	`!x. P x`
existential quantification	`∃x. P x`	`?x. P x`
Hilbert's choice operator	`@x. P x`	`@x. P x`

There are similar restrictions to constant and variable names as in SML. HOL specific: don't start variable names with an underscore

Syntax conventions

- common function syntax
  - prefix notation, e. g. `SUC x`
  - infix notation, e. g. `x + y`
  - quantifier notation, e. g. `∀x. P x` means `(∀) (λx. P x)`
- infix and quantifier notation functions can turned into prefix notation
  Example: `(+) x y` and `$+ x y` are the same as `x + y`
- quantifiers of the same type don't need to be repeated
  Example: `∀x y. P x y` is short for `∀x. ∀y. P x y`
- there is special syntax for some functions
  Example: `if c then v1 else v2` is nice syntax for `COND c v1 v2`
- associative infix operators are usually right-associative
  Example: `b1 /\ b2 /\ b3` is parsed as `b1 /\ (b2 /\ b3)`

Operator Precedence

It is easy to misjudge the binding strength of certain operators. Therefore use plenty of parenthesis.

Creating Terms

Term Parser

Use special quotation provided by `unquote`.

Use Syntax Functions

Terms are just SML values of type `term`. You can use syntax functions (usually defined in `*Syntax.sml` files) to create them.

Creating Terms II

Parser	Syntax Funs	
‘‘:bool‘‘	mk_type ("bool", []) or bool	type of Booleans
‘‘T‘‘	mk_const ("T", bool) or T	term true
‘‘~b‘‘	mk_neg (mk_var ("b", bool))	negation of Boolean var b
‘‘... /\ ...‘‘	mk_conj (..., ...)	conjunction
‘‘... \/ ...‘‘	mk_disj (..., ...)	disjunction
‘‘... ==> ...‘‘	mk_imp (..., ...)	implication
‘‘... = ...‘‘	mk_eq (..., ...)	equation
‘‘... <=> ...‘‘	mk_eq (..., ...)	equivalence
‘‘... <> ...‘‘	mk_neg (mk_eq (..., ...))	negated equation

Inference Rules for Equality

$$\frac{}{\vdash t = t}\ \mathrm{REFL}$$

$$\frac{\begin{array}{c}\Gamma \vdash s = t \\ x \textit{ not free in } \Gamma\end{array}}{\Gamma \vdash \lambda x.\ s = \lambda x. t}\ \mathrm{ABS}$$

$$\frac{\begin{array}{c}\Gamma \vdash s = t \\ \Delta \vdash u = v \\ \textit{types fit}\end{array}}{\Gamma \cup \Delta \vdash s(u) = t(v)}\ \mathrm{MK\_COMB}$$

$$\frac{\Gamma \vdash s = t}{\Gamma \vdash t = s}\ \mathrm{GSYM}$$

$$\frac{\begin{array}{c}\Gamma \vdash s = t \\ \Delta \vdash t = u\end{array}}{\Gamma \cup \Delta \vdash s = u}\ \mathrm{TRANS}$$

$$\frac{\Gamma \vdash p \Leftrightarrow q \qquad \Delta \vdash p}{\Gamma \cup \Delta \vdash q}\ \mathrm{EQ\_MP}$$

$$\frac{}{\vdash (\lambda x.\ t)x = t}\ \mathrm{BETA}$$

Inference Rules for free Variables

$$\frac{\Gamma[x_1, \ldots, x_n] \vdash p[x_1, \ldots, x_n]}{\Gamma[t_1, \ldots, t_n] \vdash p[t_1, \ldots, t_n]}\ \mathrm{INST}$$

$$\frac{\Gamma[\alpha_1, \ldots, \alpha_n] \vdash p[\alpha_1, \ldots, \alpha_n]}{\Gamma[\gamma_1, \ldots, \gamma_n] \vdash p[\gamma_1, \ldots, \gamma_n]}\ \mathrm{INST\_TYPE}$$

Inference Rules for Implication

$$\frac{\begin{array}{c}\Gamma \vdash p \Longrightarrow q \\ \Delta \vdash p\end{array}}{\Gamma \cup \Delta \vdash q}\ \text{MP, MATCH\_MP}$$

$$\frac{\Gamma \vdash p = q}{\begin{array}{c}\Gamma \vdash p \Longrightarrow q \\ \Gamma \vdash q \Longrightarrow p\end{array}}\ \text{EQ\_IMP\_RULE}$$

$$\frac{\begin{array}{c}\Gamma \vdash p \Longrightarrow q \\ \Delta \vdash q \Longrightarrow p\end{array}}{\Gamma \cup \Delta \vdash p = q}\ \text{IMP\_ANTISYM\_RULE}$$

$$\frac{\begin{array}{c}\Gamma \vdash p \Longrightarrow q \\ \Delta \vdash q \Longrightarrow r\end{array}}{\Gamma \cup \Delta \vdash p \Longrightarrow r}\ \text{IMP\_TRANS}$$

$$\frac{\Gamma \vdash p}{\Gamma - \{q\} \vdash q \Longrightarrow p}\ \text{DISCH}$$

$$\frac{\Gamma \vdash q \Longrightarrow p}{\Gamma \cup \{q\} \vdash p}\ \text{UNDISCH}$$

$$\frac{\Gamma \vdash p \Longrightarrow \mathsf{F}}{\Gamma \vdash {\sim} p}\ \text{NOT\_INTRO}$$

$$\frac{\Gamma \vdash {\sim} p}{\Gamma \vdash p \Longrightarrow \mathsf{F}}\ \text{NOT\_ELIM}$$

Inference Rules for Conjunction / Disjunction

$$\frac{\Gamma \vdash p \qquad \Delta \vdash q}{\Gamma \cup \Delta \vdash p \;\wedge\; q}\ \text{CONJ}$$

$$\frac{\Gamma \vdash p \;\wedge\; q}{\Gamma \vdash p}\ \text{CONJUNCT1}$$

$$\frac{\Gamma \vdash p \;\wedge\; q}{\Gamma \vdash q}\ \text{CONJUNCT2}$$

$$\frac{\Gamma \vdash p}{\Gamma \vdash p \;\vee\; q}\ \text{DISJ1}$$

$$\frac{\Gamma \vdash q}{\Gamma \vdash p \;\vee\; q}\ \text{DISJ2}$$

$$\frac{\begin{array}{c}\Gamma \vdash p \vee q \\ \Delta_1 \cup \{p\} \vdash r \\ \Delta_2 \cup \{q\} \vdash r\end{array}}{\Gamma \cup \Delta_1 \cup \Delta_2 \vdash r}\ \text{DISJ\_CASES}$$

Inference Rules for Quantifiers

$$\frac{\Gamma \vdash p \qquad x \text{ not free in } \Gamma}{\Gamma \vdash \forall x.\ p}\ \mathrm{GEN}$$

$$\frac{\Gamma \vdash \forall x.\ p}{\Gamma \vdash p[u/x]}\ \mathrm{SPEC}$$

$$\frac{\Gamma \vdash p[u/x]}{\Gamma \vdash \exists x.\ p}\ \mathrm{EXISTS}$$

$$\frac{\begin{array}{c}\Gamma \vdash \exists x.\ p \\ \Delta \cup \{p[u/x]\} \vdash r \\ u \text{ not free in } \Gamma, \Delta, p \text{ and } r\end{array}}{\Gamma \cup \Delta \vdash r}\ \mathrm{CHOOSE}$$

Forward Proofs

- axioms and inference rules are used to derive theorems
- this method is called **forward proof**
  - one starts with basic building blocks
  - one moves step by step forward
  - finally the theorem one is interested in is derived
- one can also implement own proof tools

Forward Proofs — Example I

Let's prove $\forall p.\ p \Longrightarrow p$.

```
val IMP_REFL_THM = let
  val tm1 = ``p:bool``;         > val tm1 = ``p``: term
  val thm1 = ASSUME tm1;        > val thm1 = [p] |- p: thm
  val thm2 = DISCH tm1 thm1;    > val thm2 = |- p ==> p: thm
in
  GEN tm1 thm2                  > val IMP_REFL_THM =
                                    |- !p. p ==> p: thm
end

fun IMP_REFL t =                > val IMP_REFL =
  SPEC t IMP_REFL_THM;              fn: term -> thm
```

Forward Proofs — Example II

Let’s prove $\forall P\ v.\ (\exists x.\ (x = v) \wedge P\ x) \iff P\ v$.

```
val tm_v = ‘‘v:’a‘‘;
val tm_P = ‘‘P:’a -> bool‘‘;
val tm_lhs = ‘‘?x. (x = v) /\ P x‘‘
val tm_rhs = mk_comb (tm_P, tm_v);

val thm1 = let
  val thm1a = ASSUME tm_rhs;
  val thm1b =
    CONJ (REFL tm_v) thm1a;
  val thm1c =
    EXISTS (tm_lhs, tm_v) thm1b
in
  DISCH tm_rhs thm1c
end
```

```
> val thm1a = [P v] |- P v: thm
> val thm1b =
    [P v] |- (v = v) /\ P v: thm
> val thm1c =
    [P v] |- ?x. (x = v) /\ P x

> val thm1 = [] |-
    P v ==> ?x. (x = v) /\ P x: thm
```

Forward Proofs — Example II cont.

```
val thm2 = let
  val thm2a =                            > val thm2a = [(u = v) /\ P u] |-
    ASSUME ‘‘(u:’a = v) /\ P u‘‘             (u = v) /\ P u: thm
  val thm2b = AP_TERM tm_P               > val thm2b = [(u = v) /\ P u] |-
    (CONJUNCT1 thm2a);                       P u <=> P v
  val thm2c = EQ_MP thm2b                > val thm2c = [(u = v) /\ P u] |-
    (CONJUNCT2 thm2a);                       P v
  val thm2d =                            > val thm2d = [?x. (x = v) /\ P x] |-
    CHOOSE (‘‘u:’a‘‘,                        P v
      ASSUME tm_lhs) thm2c
in
  DISCH tm_lhs thm2d                     > val thm2 = [] |-
end                                          ?x. (x = v) /\ P x ==> P v


val thm3 = IMP_ANTISYM_RULE thm2 thm1    > val thm3 = [] |-
                                             ?x. (x = v) /\ P x <=> P v
val thm4 = GENL [tm_P, tm_v] thm3        > val thm4 = [] |- !P v.
                                             ?x. (x = v) /\ P x <=> P v
```

Part VII

Backward Proofs

Motivation I

- let's prove `!A B. A /\ B <=> B /\ A`

```
(* Show |- A /\ B ==> B /\ A *)
val thm1a = ASSUME ``A /\ B``;
val thm1b = CONJ (CONJUNCT2 thm1a) (CONJUNCT1 thm1a);
val thm1  = DISCH ``A /\ B`` thm1b

(* Show |- B /\ A ==> A /\ B *)
val thm2a = ASSUME ``B /\ A``;
val thm2b = CONJ (CONJUNCT2 thm2a) (CONJUNCT1 thm2a);
val thm2  = DISCH ``B /\ A`` thm2b

(* Combine to get |- A /\ B <=> B /\ A *)
val thm3  = IMP_ANTISYM_RULE thm1 thm2

(* Add quantifiers *)
val thm4  = GENL [``A:bool``, ``B:bool``] thm3
```

- this is how you write down a proof
- for finding a proof it is however often useful to think **backwards**

Motivation II - thinking backwards

- we want to prove
  - `!A B. A /\ B <=> B /\ A`
- all-quantifiers can easily be added later, so let's get rid of them
  - `A /\ B <=> B /\ A`
- now we have an equivalence, let's show 2 implications
  - `A /\ B ==> B /\ A`
  - `B /\ A ==> A /\ B`
- we have an implication, so we can use the precondition as an assumption
  - using `A /\ B` show `B /\ A`
  - `A /\ B ==> B /\ A`

Motivation III - thinking backwards

- we have a conjunction as assumption, let's split it
  - using `A` and `B` show `B /\ A`
  - `A /\ B ==> B /\ A`
- we have to show a conjunction, so let's show both parts
  - using `A` and `B` show `B`
  - using `A` and `B` show `A`
  - `A /\ B ==> B /\ A`
- the first two proof obligations are trivial
  - `A /\ B ==> B /\ A`
- . . .
- we are done

Motivation IV

- common practise
  - think backwards to find proof
  - write found proof down in forward style
- often switch between backward and forward style within a proof
  Example: induction proof
  - backward step: induct on ...
  - forward steps: prove base case and induction case
- whether to use forward or backward proofs depend on
  - support by the interactive theorem prover you use
    - HOL 4 and close family: emphasis on backward proof
    - Isabelle/HOL: emphasis on forward proof
    - Coq : emphasis on backward proof
  - your way of thinking
  - the theorem you try to prove

HOL Implementation of Backward Proofs

- in HOL
  - proof tactics / backward proofs used for most user-level proofs
  - forward proofs used usually for writing automation
- backward proofs are implemented by **tactics** in HOL
  - decomposition into subgoals implemented in SML
  - SML datastructures used to keep track of all open subgoals
  - forward proof used to construct theorems
- to understand backward proofs in HOL we need to look at
  - `goal` — SML datatype for proof obligations
  - `goalStack` — library for keeping track of goals
  - `tactic` — SML type for functions performing backward proofs

Goals

- goals represent proof obligations, i. e. theorems we need/want to prove
- the SML type goal is an abbreviation for term list * term
- the goal ([asm_1, ..., asm_n], c) records that we need/want to prove the theorem {asm_1, ..., asm_n} |- c

Example Goals

<table>
<tr><th>Goal</th><th>Theorem</th></tr>
<tr><td>([``A``, ``B``], ``A /\ B``)</td><td>{A, B} |- A /\ B</td></tr>
<tr><td>([``B``, ``A``], ``A /\ B``)</td><td>{A, B} |- A /\ B</td></tr>
<tr><td>([``B /\ A``], ``A /\ B``)</td><td>{B /\ A} |- A /\ B</td></tr>
<tr><td>([], ``(B /\ A) ==> (A /\ B)``)</td><td>|- (B /\ A) ==> (A /\ B)</td></tr>
</table>

Tactics

- the SML type `tactic` is an abbreviation for the type `goal -> goal list * validation`
- `validation` is an abbreviation for `thm list -> thm`
- given a goal, a tactic
  - decides into which subgoals to decompose the goal
  - returns this list of subgoals
  - returns a validation that
    - given a list of theorems for the computed subgoals
    - produces a theorem for the original goal
- special case: empty list of subgoals
  - the validation (given `[]`) needs to produce a theorem for the goal
- notice: a tactic might be invalid

Tactic Example — `CONJ_TAC`

$$\frac{\Gamma \vdash p \qquad \Delta \vdash q}{\Gamma \cup \Delta \vdash p \ \wedge \ q}\ \mathrm{CONJ}$$

$$\frac{\begin{array}{c}\texttt{t} \equiv \texttt{conj1 /\textbackslash{} conj2} \\ \texttt{asl} \vdash \texttt{conj1} \qquad \texttt{asl} \vdash \texttt{conj2}\end{array}}{\texttt{asl} \vdash \texttt{t}}$$

```
val CONJ_TAC: tactic = fn (asl, t) =>
   let
      val (conj1, conj2) = dest_conj t
   in
      ([(asl, conj1), (asl, conj2)],
       fn [th1, th2] => CONJ th1 th2 | _ => raise Match)
   end
   handle HOL_ERR _ => raise ERR "CONJ_TAC" ""
```

Tactic Example — EQ_TAC

$$\frac{\begin{array}{c}\Gamma \vdash p \Longrightarrow q \\ \Delta \vdash q \Longrightarrow p\end{array}}{\Gamma \cup \Delta \vdash p = q}\ \mathrm{IMP\_ANTISYM\_RULE}$$

$$\begin{array}{c}\texttt{t} \equiv \texttt{lhs = rhs} \\ \dfrac{\begin{array}{c}\texttt{asl} \vdash \texttt{lhs ==> rhs} \\ \texttt{asl} \vdash \texttt{rhs ==> lhs}\end{array}}{\texttt{asl} \vdash \texttt{t}}\end{array}$$

```
val EQ_TAC: tactic = fn (asl, t) =>
   let
      val (lhs, rhs) = dest_eq t
   in
      ([(asl, mk_imp (lhs, rhs)), (asl, mk_imp (rhs, lhs))],
       fn [th1, th2] => IMP_ANTISYM_RULE th1 th2
        | _          => raise Match)
   end
   handle HOL_ERR _ => raise ERR "EQ_TAC" ""
```

proofManagerLib / goalStack

- the `proofManagerLib` keeps track of open goals
- it uses `goalStack` internally
- important commands
  - **g** — set up new goal
  - **e** — expand a tactic
  - **p** — print the current status
  - **top_thm** — get the proved thm at the end

Tactic Proof Example I

Previous Goalstack

```
-
```

User Action

```
g ‘!A B. A /\ B <=> B /\ A‘;
```

New Goalstack

```
Initial goal:

!A B. A /\ B <=> B /\ A

: proof
```

Tactic Proof Example II

Previous Goalstack

```
Initial goal:

!A B. A /\ B <=> B /\ A

: proof
```

User Action

```
e GEN_TAC;
e GEN_TAC;
```

New Goalstack

```
A /\ B <=> B /\ A

: proof
```

Tactic Proof Example III

Previous Goalstack

```
A /\ B <=> B /\ A


: proof
```

User Action

```
e EQ_TAC;
```

New Goalstack

```
B /\ A ==> A /\ B


A /\ B ==> B /\ A


: proof
```

Tactic Proof Example IV

Previous Goalstack

```
B /\ A ==> A /\ B


A /\ B ==> B /\ A : proof
```

User Action

```
e STRIP_TAC;
```

New Goalstack

```
B /\ A
------------------------------------
  0.  A
  1.  B
```

Tactic Proof Example V

Previous Goalstack

```
B /\ A
------------------------------------
  0.  A
  1.  B
```

User Action

```
e CONJ_TAC;
```

New Goalstack

```
A
------------------------------------
  0.  A
  1.  B

B
------------------------------------
  0.  A
  1.  B
```

Tactic Proof Example VI

Previous Goalstack

```
A
------------------------------------
  0.  A
  1.  B

B
------------------------------------
  0.  A
  1.  B
```

User Action

```
e (ACCEPT_TAC (ASSUME ‘‘B:bool‘‘));
e (ACCEPT_TAC (ASSUME ‘‘A:bool‘‘));
```

New Goalstack

```
B /\ A ==> A /\ B


: proof
```

Tactic Proof Example VII

Previous Goalstack

```
B /\ A ==> A /\ B


: proof
```

User Action

```
e STRIP_TAC;
e (ASM_REWRITE_TAC[]);
```

New Goalstack

```
Initial goal proved.
|- !A B. A /\ B <=> B /\ A:
   proof
```

Tactic Proof Example VIII

Previous Goalstack

```
Initial goal proved.
|- !A B. A /\ B <=> B /\ A:
   proof
```

User Action

```
val thm = top_thm();
```

Result

```
val thm =
   |- !A B. A /\ B <=> B /\ A:
   thm
```

Tactic Proof Example IX

Combined Tactic

```
val thm = prove (``!A B. A /\ B <=> B /\ A``,
  GEN_TAC >> GEN_TAC >>
  EQ_TAC >| [
    STRIP_TAC >>
    STRIP_TAC >| [
      ACCEPT_TAC (ASSUME ``B:bool``),
      ACCEPT_TAC (ASSUME ``A:bool``)
    ],

    STRIP_TAC >>
    ASM_REWRITE_TAC[]
  ]);
```

Result

```
val thm =
   |- !A B. A /\ B <=> B /\ A:
   thm
```

Tactic Proof Example X

Cleaned-up Tactic

```
val thm = prove (``!A B. A /\ B <=> B /\ A``,
  REPEAT GEN_TAC >>
  EQ_TAC >> (
    REPEAT STRIP_TAC >>
    ASM_REWRITE_TAC []
  ));
```

Result

```
val thm =
   |- !A B. A /\ B <=> B /\ A:
   thm
```

Summary Backward Proofs

- in HOL most user-level proofs are tactic-based
  - automation often written in forward style
  - low-level, basic proofs written in forward style
  - nearly everything else is written in backward (tactic) style
- there are **many** different tactics
- in the lecture only the most basic ones will be discussed
- you need to learn about tactics on your own
  - good starting point: `Quick` manual
  - learning finer points takes a lot of time
  - exercises require you to read up on tactics
- often there are many ways to prove a statement, which tactics to use depends on
  - personal way of thinking
  - personal style and preferences
  - maintainability, clarity, elegance, robustness
  - . . .

Part VIII

Basic Tactics

Syntax of Tactics in HOL

- originally tactics were written all in capital letters with underscores
  Example: `ALL_TAC`
- since 2010 more and more tactics have overloaded lower-case syntax
  Example: `all_tac`
- sometimes, the lower-case version is shortened
  Example: `REPEAT`, `rpt`
- sometimes, there is special syntax
  Example: `THEN`, `\\`, `>>`
- which one to use is mostly a matter of personal taste
  - all-capital names are hard to read and type
  - however, not for all tactics there are lower-case versions
  - mixed lower- and upper-case tactics are even harder to read
  - often shortened lower-case name is not *speaking*

In the lecture we will use mostly the old-style names.

Some Basic Tactics

GEN_TAC	remove outermost all-quantifier
DISCH_TAC	move antecedent of goal into assumptions
CONJ_TAC	splits conjunctive goal
STRIP_TAC	splits on outermost connective (combination of GEN_TAC, CONJ_TAC, DISCH_TAC, . . .)
DISJ1_TAC	selects left disjunct
DISJ2_TAC	selects right disjunct
EQ_TAC	reduce Boolean equality to implications
ASSUME_TAC thm	add theorem to list of assumptions
EXISTS_TAC term	provide witness for existential goal

Tacticals

- tacticals are SML functions that combine tactics to form new tactics
- common workflow
  - develop large tactic interactively
  - using `goalStack` and editor support to execute tactics one by one
  - combine tactics manually with tacticals to create larger tactics
  - finally end up with one large tactic that solves your goal
  - use `prove` or `store_thm` instead of `goalStack`
- make sure to clearly mark proof structure by e. g.
  - use indentation
  - use parentheses
  - use appropriate connectives
  - ...
- goalStack commands like `e` or `g` should not appear in your final proof

Some Basic Tacticals

tac1 >> tac2	THEN, \\	applies tactics in sequence
tac >| tacL	THENL	applies list of tactics to subgoals
tac1 >- tac2	THEN1	applies tac2 to the first subgoal of tac1
REPEAT tac	rpt	repeats tac until it fails
NTAC n tac		apply tac n times
REVERSE tac	reverse	reverses the order of subgoals
tac1 ORELSE tac2		applies tac1 only if tac2 fails
TRY tac		do nothing if tac fails
ALL_TAC	all_tac	do nothing
NO_TAC		fail

Basic Rewrite Tactics

- (equational) rewriting is at the core of HOL's automation
- we will discuss it in detail later
- details complex, but basic usage is straightforward
  - given a theorem `rewr_thm` of form `|- P x = Q x` and a term `t`
  - rewriting `t` with `rewr_thm` means
  - replacing each occurrence of a term `P c` for some `c` with `Q c` in `t`
- warning: rewriting may loop
  Example: rewriting with theorem `|- X <=> (X /\ T)`

`REWRITE_TAC` thms	rewrite goal using equations found in given list of theorems
`ASM_REWRITE_TAC` thms	in addition use assumptions
`ONCE_REWRITE_TAC` thms	rewrite once in goal using equations
`ONCE_ASM_REWRITE_TAC` thms	rewrite once using assumptions

Case-Split and Induction Tactics

Induct_on ‘term‘	induct on term
Induct	induct on all-quantor
Cases_on ‘term‘	case-split on term
Cases	case-split on all-quantor
MATCH_MP_TAC thm	apply rule
IRULE_TAC thm	generalised apply rule

Assumption Tactics

POP_ASSUM thm-tac	use and remove first assumption common usage POP_ASSUM MP_TAC
PAT_ASSUM term thm-tac also PAT_X_ASSUM term thm-tac	use (and remove) first assumption matching pattern
WEAKEN_TAC term-pred	removes first assumption satisfying predicate

Decision Procedure Tactics

- decision procedures try to solve the current goal completely
- they either succeed of fail
- no partial progress
- decision procedures vital for automation

`TAUT_TAC`	propositional logic tautology checker
`DECIDE_TAC`	linear arithmetic for `num`
`METIS_TAC` thms	first order prover
`numLib.ARITH_TAC`	Presburger arithmetic
`intLib.ARITH_TAC`	uses Omega test

Subgoal Tactics

- it is vital to structure your proofs well
  - improved maintainability
  - improved readability
  - improved reusability
  - saves time in medium-run
- therefore, use many small lemmata
- also, use many explicit subgoals

‘term-frag‘ `by` tac	show term with tac and add it to assumptions
‘term-frag‘ `sufficies_by` tac	show it sufficies to prove term

Term Fragments / Term Quotations

- notice that `by` and `sufficies_by` take **term fragments**
- term fragments are also called **term quotations**
- they represent (partially) unparsed terms
- parsing takes time place during execution of tactic in context of goal
- this helps to avoid type annotations
- however, this means syntax errors show late as well
- the library **Q** defines many tactics using term fragments

Importance of Exercises

- here many tactics are presented in a very short amount of time
- there are many, many more important tactics out there
- few people can learn a programming language just by reading manuals
- similar few people can learn HOL just by reading and listening
- you should write your own proofs and play around with these tactics
- solving the exercises is highly recommended
  (and actually required if you want credits for this course)

Tactical Proof - Example I - Slide 1

- we want to prove `!l. LENGTH (APPEND l l) = 2 * LENGTH l`
- first step: set up goal on `goalStack`
- at same time start writing proof script

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
```

Actions

- run g ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``
- this is done by hol-mode
- move cursor inside term and press `M-h g`
  (menu-entry `HOL - Goalstack - New goal`)

Tactical Proof - Example I - Slide 2

Current Goal

```
!l. LENGTH (l ++ l) = 2 * LENGTH l
```

- the outermost connective is an all-quantor
- let's get rid of it via `GEN_TAC`

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (l ++ l) = 2 * LENGTH l``,
GEN_TAC
```

Actions

- run `e GEN_TAC`
- this is done by hol-mode
- mark line with `GEN_TAC` and press `M-h e` (menu-entry `HOL - Goalstack - Apply tactic`)

Tactical Proof - Example I - Slide 3

Current Goal

```
LENGTH (l ++ l) = 2 * LENGTH l
```

- LENGTH of APPEND can be simplified
- let's search an appropriate lemma with DB.match

Actions

- run `DB.print_match [] ``LENGTH (_ ++ _)``
- this is done via hol-mode
- press M-h m and enter term pattern
  (menu-entry HOL - Misc - DB match)
- this finds the theorem listTheory.LENGTH_APPEND
  `|- !l1 l2. LENGTH (l1 ++ l2) = LENGTH l1 + LENGTH l2`

Tactical Proof - Example I - Slide 4

Current Goal

```
LENGTH (l ++ l) = 2 * LENGTH l
```

- let's rewrite with found theorem `listTheory.LENGTH_APPEND`

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND]
```

Actions

- connect the new tactic with tactical `>>` (`THEN`)
- use hol-mode to expand the new tactic

Tactical Proof - Example I - Slide 5

Current Goal

```
LENGTH l + LENGTH l = 2 * LENGTH l
```

- let's search a theorem for simplifying `2 * LENGTH l`
- prepare for extending the previous rewrite tactic

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND]
```

Actions

- `DB.match` finds theorem `arithmeticTheory.TIMES2`
- press `M-h b` and undo last tactic expansion
  (menu-entry `HOL - Goalstack - Back up`)

Tactical Proof - Example I - Slide 6

Current Goal

```
LENGTH (l ++ l) = 2 * LENGTH l
```

- extend the previous rewrite tactic
- finish proof

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ‘‘!l. LENGTH (APPEND l l) = 2 * LENGTH l‘‘,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Actions

- add `TIMES2` to the list of theorems used by rewrite tactic
- use hol-mode to expand the extended rewrite tactic
- goal is solved, so let’s add closing parenthesis and semicolon

Tactical Proof - Example I - Slide 7

- we have a finished tactic proving our goal
- notice that GEN_TAC is not needed
- let’s polish the proof script

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ‘‘!l. LENGTH (APPEND l l) = 2 * LENGTH l‘‘,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Polished Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ‘‘!l. LENGTH (APPEND l l) = 2 * LENGTH l‘‘,
REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Tactical Proof - Example II - Slide 1

- let's prove something slightly more complicated
- drop old goal by pressing `M-h d`
  (menu-entry `HOL - Goalstack - Drop goal`)
- set up goal on `goalStack` (`M-h g`)
- at same time start writing proof script

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (``!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>
  ~(ALL_DISTINCT (l1 ++ l2 ++ l3))``,
```

Tactical Proof - Example II - Slide 2

Current Goal

```
!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  x1 <= x2 /\ x2 <= x3 /\ x3 <= SUC x1 ==>
  ~ALL_DISTINCT (l1 ++ l2 ++ l3)
```

- let's strip the goal

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>
  ~(ALL_DISTINCT (l1 ++ l2 ++ l3))‘‘,
REPEAT STRIP_TAC
```

Tactical Proof - Example II - Slide 2

Current Goal

```
!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  x1 <= x2 /\ x2 <= x3 /\ x3 <= SUC x1 ==>
  ~ALL_DISTINCT (l1 ++ l2 ++ l3)
```

- let's strip the goal

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ‘‘!l. LENGTH (APPEND l l) = 2 * LENGTH l‘‘,
REPEAT STRIP_TAC
```

Actions

- add `REPEAT STRIP_TAC` to proof script
- expand this tactic using hol-mode

Tactical Proof - Example II - Slide 3

Current Goal

```
F
------------------------------------
  0.  MEM x1 l1        4.  x2 <= x3
  1.  MEM x2 l2        5.  x3 <= SUC x1
  2.  MEM x3 l3        6.  ALL_DISTINCT (l1 ++ l2 ++ l3)
  3.  x1 <= x2
```

- oops, we did too much, we would like to keep `ALL_DISTINCT` in goal

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘...‘‘,
REPEAT GEN_TAC >> STRIP_TAC
```

Actions

- undo `REPEAT STRIP_TAC` (`M-h b`)
- expand more fine-tuned strip tactic

Tactical Proof - Example II - Slide 4

Current Goal

```
~ALL_DISTINCT (l1 ++ l2 ++ l3)
------------------------------------
  0.  MEM x1 l1        3.  x1 <= x2
  1.  MEM x2 l2        4.  x2 <= x3
  2.  MEM x3 l3        5.  x3 <= SUC x1
```

- now let's simplify `ALL_DISTINCT`
- search suitable theorems with `DB.match`
- use them with rewrite tactic

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘...‘‘,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND]
```

Tactical Proof - Example II - Slide 5

Current Goal

```
~((ALL_DISTINCT l1 /\ ALL_DISTINCT l2 /\ !e. MEM e l1 ==> ~MEM e l2) /\
  ALL_DISTINCT l3 /\ !e. MEM e l1 \/ MEM e l2 ==> ~MEM e l3)
------------------------------------
  0.  MEM x1 l1        3.  x1 <= x2
  1.  MEM x2 l2        4.  x2 <= x3
  2.  MEM x3 l3        5.  x3 <= SUC x1
```

- from assumptions 3, 4 and 5 we know `x2 = x1 \/ x2 = x3`
- let’s deduce this fact by `DECIDE_TAC`

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘...‘‘,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>
‘(x2 = x1) \/ (x2 = x3)‘ by DECIDE_TAC
```

Tactical Proof - Example II - Slide 6

Current Goals — 2 subgoals, one for each disjunct

```
~((ALL_DISTINCT l1 /\ ALL_DISTINCT l2 /\ !e. MEM e l1 ==> ~MEM e l2) /\
  ALL_DISTINCT l3 /\ !e. MEM e l1 \/ MEM e l2 ==> ~MEM e l3)
------------------------------------
  0.  MEM x1 l1         4.  x2 <= x3
  1.  MEM x2 l2         5.  x3 <= SUC x1
  2.  MEM x3 l3         6a. x2 = x1
  3.  x1 <= x2          6b. x2 = x3
```

- both goals are easily solved by first-order reasoning
- let's use `METIS_TAC[]` for both subgoals

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (‘‘...‘‘,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>
‘(x2 = x1) \/ (x2 = x3)‘ by DECIDE_TAC >> (
  METIS_TAC[]
));
```

Tactical Proof - Example II - Slide 7

Finished Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (
``!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>
  ~(ALL_DISTINCT (l1 ++ l2 ++ l3))``,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>
`(x2 = x1) \/ (x2 = x3)` by DECIDE_TAC >> (
  METIS_TAC[]
));
```

- notice that proof structure is explicit
- parentheses and indentation used to mark new subgoals

Part IX

Induction Proofs

Mathematical Induction

- mathematical (a. k. a. natural) induction principle:
  If a property P holds for 0 and $P(n)$ implies $P(n+1)$ for all n, then $P(n)$ holds for all n.
- HOL is expressive enough to encode this principle as a theorem.

```
|- !P. P 0 /\ (!n. P n ==> P (SUC n)) ==> !n. P n
```

- Performing mathematical induction in HOL means applying this theorem (e. g. via `HO_MATCH_MP_TAC`)
- there are many similarish induction theorems in HOL
- Example: complete induction principle

```
|- !P. (!n. (!m. m < n ==> P m) ==> P n) ==> !n. P n
```

Structural Induction Theorems

- **structural induction** theorems are an important special form of induction theorems
- they describe performing induction on the structure of a datatype
- Example: `|- !P. P [] /\ (!t. P t ==> !h. P (h::t)) ==> !l. P l`
- structural induction is used very frequently in HOL
- for each algabraic datatype, there is an induction theorem

Other Induction Theorems

- there are many induction theorems in HOL
  - datatype definitions lead to induction theorems
  - recursive function definitions produce corresponding induction theorems
  - recursive relation definitions give rise to induction theorems
  - many are manually defined
- Examples

```
|- !P. P [] /\ (!l. P l ==> !x. P (SNOC x l)) ==> !l. P l

|- !P. P FEMPTY /\
      (!f. P f ==> !x y. x NOTIN FDOM f ==> P (f |+ (x,y))) ==> !f. P f

|- !P. P {} /\
      (!s. FINITE s /\ P s ==> !e. e NOTIN s ==> P (e INSERT s)) ==>
      !s. FINITE s ==> P s

|- !R P. (!x y. R x y ==> P x y) /\ (!x y z. P x y /\ P y z ==> P x z) ==>
      !u v. R^+ u v ==> P u v
```

Induction (and Case-Split) Tactics

- the tactic `Induct` (or `Induct_on`) usually used to start induction proofs
- it looks at the type of the quantifier (or its argument) and applies the default induction theorem for this type
- this is usually what one needs
- other (non default) induction theorems can be applied via `INDUCT_THEN` or `HO_MATCH_MP_TAC`
- similarish `Cases_on` picks and applies default case-split theorems

Induction Proof - Example I - Slide 1

- let's prove via induction
  `!l1 l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1`
- we set up the goal and start and induction proof on `l1`

Proof Script

```
val REVERSE_APPEND = prove (
``!l1 l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1``,
Induct
```

Induction Proof - Example I - Slide 2

- the induction tactic produced two cases
- base case:

  ```
  !l2. REVERSE ([] ++ l2) = REVERSE l2 ++ REVERSE []
  ```

- induction step:

  ```
  !h l2. REVERSE (h::l1 ++ l2) = REVERSE l2 ++ REVERSE (h::l1)
  ------------------------------------------------------------
  !l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1
  ```

- both goals can be easily proved by rewriting

Proof Script

```
val REVERSE_APPEND = prove (‘‘
!l1 l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1‘‘,
Induct >| [
  REWRITE_TAC[REVERSE_DEF, APPEND, APPEND_NIL],
  ASM_REWRITE_TAC[REVERSE_DEF, APPEND, APPEND_ASSOC]
]);
```

Induction Proof - Example II - Slide 2

- let’s prove via induction
  `!l. REVERSE (REVERSE l) = l`
- we set up the goal and start and induction proof on `l`

Proof Script

```
val REVERSE_REVERSE = prove (
‘‘!l. REVERSE (REVERSE l) = l‘‘,
Induct
```

Induction Proof - Example II - Slide 2

- the induction tactic produced two cases
- base case:

  ```
  REVERSE (REVERSE []) = []
  ```

- induction step:

  ```
  !h. REVERSE (REVERSE (h::l1)) = h::l1
  ----------------------------------------------
  REVERSE (REVERSE l) = l
  ```

- again both goals can be easily proved by rewriting

Proof Script

```
val REVERSE_REVERSE = prove (
``!l. REVERSE (REVERSE l) = l``,
Induct >| [
  REWRITE_TAC[REVERSE_DEF],
  ASM_REWRITE_TAC[REVERSE_DEF, REVERSE_APPEND, APPEND]
]);
```

Part X

Basic Definitions

Definitional Extensions

- there are **conservative definition principles** for types and constants
- conservative means that all theorems that can be proved in extended theory can also be proved in original one
- however, such extensions make the theory more comfortable
- definitions introduce no new inconsistencies
- the HOL community has a very strong tradition of a purely definitional approach

Axiomatic Extensions

- axioms are a different approach
- they allow postulating arbitrary properties, i. e. extending the logic with arbitrary theorems
- this approach might introduce new inconsistencies
- in HOL axioms are very rarely needed
- using definitions is often considered more elegant
- it is hard to keep track of axioms
- use axioms only if you really know what you are doing

Oracles

- oracles are families of axioms
- however, they are used differently than axioms
- they are used to enable usage of external tools and knowledge
- you might want to use an external automated prover
- this external tool acts as an oracle
  - it provides answers
  - it does not explain or justify these answers
- you don't know, whether this external tool might be buggy
- all theorems proved via it are tagged with a special oracle-tag
- tags are propagated
- this allows keeping track of everything depending on the correctness of this tool

Oracles II

- Common oracle-tags
  - `DISK_THM` — theorem was written to disk and read again
  - `HolSatLib` — proved by MiniSat
  - `HolSmtLib` — proved by external SMT solver
  - `fast_proof` — proof was skipped to compile a theory rapidly
  - `cheat` — we cheated :-)
- cheating via e. g. the `cheat` tactic means skipping proofs
- it can be helpful during proof development
  - test whether some lemmata allow you finishing the proof
  - skip lengthy but boring cases and focus on critical parts first
  - experiment with exact form of invariants
  - . . .
- cheats should be removed reasonable quickly
- HOL warns about cheats and skipped proofs

Pitfalls of Definitional Approach

- definitions can't introduce new inconsistencies
- they force you to state all assumed properties at one location
- however, you still need to be careful
- Is your definition really expressing what you had in mind ?
- Does your formalisation correspond to the real world artefact ?
- How can you convince others that this is the case ?
- we will discuss methods to deal with this later in this course
    - formal sanity
    - conformance testing
    - code review
    - comments, good names, clear coding style
    - . . .
- this is highly complex and needs a lot of effort in general

Specifications

- HOL allows to introduce new constants with certain properties, provided the existence of such constants has been shown

Specification of EVEN and ODD

```
> EVEN_ODD_EXISTS
val it = |- ?even odd. even 0 /\ ~odd 0 /\ (!n. even (SUC n) <=> odd n) /\
                       (!n. odd (SUC n) <=> even n)

> val EO_SPEC = new_specification ("EO_SPEC", ["EVEN", "ODD"], EVEN_ODD_EXISTS);
val EO_SPEC = |- EVEN 0 /\ ~ODD 0 /\ (!n. EVEN (SUC n) <=> ODD n) /\
                 (!n. ODD (SUC n) <=> EVEN n)
```

- `new_specification` is a convenience wrapper
  - it uses existential quantification instead of Hilbert's choice
  - deals with pair syntax
  - stores resulting definitions in theory
- `new_specification` captures the underlying principle nicely

Definitions

- special case: new constant defined by equality

Specification with Equality

```
> double_EXISTS
val it =
|- ?double. (!n. double n = (n + n))

> val double_def = new_specification ("double_def", ["double"], double_EXISTS);
val double_def =
   |- !n. double n = n + n
```

- there is a specialised methods for such non-recursive definitions

Non Recursive Definitions

```
> val DOUBLE_DEF = new_definition ("DOUBLE_DEF", ``DOUBLE n = n + n``)
val DOUBLE_DEF =
   |- !n. DOUBLE n = n + n
```

Restrictions for Definitions

- all variables occurring on right-hand-side (rhs) need to be arguments
  - e. g. new_definition (..., ``F n = n + m``) fails
  - m is free on rhs
- all type variables occurring on rhs need to occur on lhs
  - e. g. new_definition ("IS_FIN_TY",
    ``IS_FIN_TY = FINITE (UNIV : 'a set)``) fails
  - IS_FIN_TY would lead to inconsistency
  - |- FINITE (UNIV : bool set)
  - |- ~FINITE (UNIV : num set)

  -
    ```
    T <=> FINITE (UNIV:bool set) <=>
    IS_FIN_TY <=>
    FINITE (UNIV:num set) <=> F
    ```
  - therefore, such definitions can't be allowed

Underspecified Functions

- function specification do not need to define the function precisely
- multiple different functions satisfying one spec are possible
- functions resulting from such specs are called **underspecified**
- underspecified functions are still total, one just lacks knowledge
- one common application: modelling **partial functions**
  - functions like e. g. `HD` and `TL` are total
  - they are defined for empty lists
  - however, it is not specified, which value they have for empty lists
  - only known: `HD [] = HD []` and `TL [] = TL []`

```
val MY_HD_EXISTS = prove (``?hd. !x xs. (hd (x::xs) = x)``, ...);
val MY_HD_SPEC =
  new_specification ("MY_HD_SPEC", ["MY_HD"], MY_HD_EXISTS)
```

Primitive Type Definitions

- HOL allows introducing non-empty subtypes of existing types
- a predicate `P : ty -> bool` describes a subset of an existing type `ty`
- `ty` may contain type variables
- only **non-empty** types are allowed
- therefore a non-emptyness proof `ex-thm` of form `?e. P e` is needed
- `new_type_definition (op-name, ex-thm)` then introduces a new type `op-name` specified by `P`

Primitive Type Definitions - Example 1

- lets try to define a type `dlist` of lists containing no duplicates
- predicate `ALL_DISTINCT : 'a list -> bool` is used to define it
- easy to prove theorem dlist_exists: `|- ?l. ALL_DISTINCT l`
- `val dlist_TY_DEF = new_type_definitions("dlist", dlist_exists)` defines a new type `'a dlist` and returns a theorem

```
|- ?(rep :'a dlist -> 'a list).
     TYPE_DEFINITION ALL_DISTINCT rep
```

- `rep` is a function taking a `'a dlist` to the list representing it
  - `rep` is injective
  - a list satisfies `ALL_DISTINCT` iff there is a corresponding `dlist`

Primitive Type Definitions - Example 2

- define_new_type_bijections can be used to define bijections between old and new type

```
> define_new_type_bijections {name="dlist_tybij", ABS="abs_dlist",
    REP="rep_dlist", tyax=dlist_TY_DEF}

val it =
   |- (!a. abs_dlist (rep_dlist a) = a) /\
      (!r. ALL_DISTINCT r <=> (rep_dlist (abs_dlist r) = r))
```

- other useful theorems can be automatically proved by
  - prove_abs_fn_one_one
  - prove_abs_fn_onto
  - prove_rep_fn_one_one
  - prove_rep_fn_onto

Primitive Definition Principles Summary

- primitive definition principles are easily explained
- they lead to conservative extensions
- however, they are cumbersome to use
- LCF approach allows implementing more convenient definition tools
  - `Datatype` package
  - `TFL` (Total Functional Language) package
  - `IndDef` (Inductive Definition) package
  - `quotientLib` Quotient Types Library
  - ...

Functional Programming

- the Datatype package allows to define datatypes conveniently
- the TFL package allows to define (mutually recursive) functions
- the EVAL conversion allows evaluating those definitions
- this gives many HOL developments the feeling of a functional program
- there is really a close connection between functional programming a definitions in HOL
  - functional programming design principles apply
  - EVAL is a great way to test quickly, whether your definitions are working as intended

Functional Programming Example

```
> Datatype ‘mylist = E | L ’a mylist‘
val it = (): unit

> Define ‘(mylen E = 0) /\ (mylen (L x xs) = SUC (mylen xs))‘
Definition has been stored under "mylen_def"
val it =
   |- (mylen E = 0) /\ !x xs. mylen (L x xs) = SUC (mylen xs):
   thm

> EVAL ‘‘mylen (L 2 (L 3 (L 1 E)))‘‘
val it =
   |- mylen (L 2 (L 3 (L 1 E))) = 3:
   thm
```

Datatype Package

- the `Datatype` package allows to define SML style datatypes easily
- there is support for
  - algebraic datatypes
  - record types
  - mutually recursive types
  - ...
- many constants are automatically introduced
  - constructors
  - case-split constant
  - size function
  - field-update and accessor functions for records
  - ...
- many theorems are derived and stored in current theory
  - injectivity and distinctness of constructors
  - nchotomy and structural induction theorems
  - rewrites for case-split, size and record update functions
  - ...

Datatype Package - Example I

Tree Datatype in SML

```
datatype ('a,'b) btree =   Leaf of 'a
                         | Node of ('a,'b) btree * 'b * ('a,'b) btree
```

Tree Datatype in HOL

```
Datatype `btree =   Leaf 'a
                  | Node btree 'b btree`
```

Tree Datatype in HOL — Deprecated Syntax

```
Hol_datatype `btree =   Leaf of 'a
                      | Node of btree => 'b => btree`
```

Datatype Package - Example I - Derived Theorems 1

btree_distinct

```
|- !a2 a1 a0 a. Leaf a <> Node a0 a1 a2
```

btree_11

```
|- (!a a'. (Leaf a = Leaf a') <=> (a = a')) /\
   (!a0 a1 a2 a0' a1' a2'.
      (Node a0 a1 a2 = Node a0' a1' a2') <=>
      (a0 = a0') /\ (a1 = a1') /\ (a2 = a2'))
```

btree_nchotomy

```
|- !bb. (?a. bb = Leaf a) \/ (?b b1 b0. bb = Node b b1 b0)
```

btree_induction

```
|- !P. (!a. P (Leaf a)) /\
       (!b b0. P b /\ P b0 ==> !b1. P (Node b b1 b0)) ==>
       !b. P b
```

Datatype Package - Example I - Derived Theorems 2

btree_size_def

```
|- (!f f1 a. btree_size f f1 (Leaf a) = 1 + f a) /\
   (!f f1 a0 a1 a2.
     btree_size f f1 (Node a0 a1 a2) =
     1 + (btree_size f f1 a0 + (f1 a1 + btree_size f f1 a2)))
```

bbtree_case_def

```
|- (!a f f1. btree_CASE (Leaf a) f f1 = f a) /\
   (!a0 a1 a2 f f1. btree_CASE (Node a0 a1 a2) f f1 = f1 a0 a1 a2)
```

btree_case_cong

```
|- !M M' f f1.
     (M = M') /\ (!a. (M' = Leaf a) ==> (f a = f' a)) /\
     (!a0 a1 a2.
        (M' = Node a0 a1 a2) ==> (f1 a0 a1 a2 = f1' a0 a1 a2)) ==>
     (btree_CASE M f f1 = btree_CASE M' f' f1')
```

Datatype Package - Example II

Enumeration type in SML

```
datatype my_enum = E1 | E2 | E3
```

Enumeration type in HOL

```
Datatype ‘my_enum = E1 | E2 | E3‘
```

Datatype Package - Example II - Derived Theorems

my_enum_nchotomy

```
|- !P. P E1 /\ P E2 /\ P E3 ==> !a. P a
```

my_enum_distinct

```
|- E1 <> E2 /\ E1 <> E3 /\ E2 <> E3
```

my_enum2num_thm

```
|- (my_enum2num E1 = 0) /\ (my_enum2num E2 = 1) /\ (my_enum2num E3 = 2)
```

my_enum2num_num2my_enum

```
|- !r. r < 3 <=> (my_enum2num (num2my_enum r) = r)
```

Datatype Package - Example III

Record type in SML

```
type rgb = { r : int, g : int, b : int }
```

Record type in HOL

```
Datatype ‘rgb = <| r : num; g : num; b : num |>‘
```

Datatype Package - Example III - Derived Theorems

rgb_component_equality

```
|- !r1 r2. (r1 = r2) <=>
           (r1.r = r2.r) /\ (r1.g = r2.g) /\ (r1.b = r2.b)
```

rgb_nchotomy

```
|- !rr. ?n n0 n1. rr = rgb n n0 n1
```

rgb_r_fupd

```
|- !f n n0 n1. rgb n n0 n1 with r updated_by f = rgb (f n) n0 n1
```

rgb_updates_eq_literal

```
|- !r n1 n0 n.
     r with <|r := n1; g := n0; b := n|> = <|r := n1; g := n0; b := n|>
```

Datatype Package - Example IV

- nested record types are not allowed
- however, mutual recursive types can mitigate this restriction

Filesystem Datatype in SML

```
datatype file = Text of string
              | Dir of {owner : string ,
                        files : (string * file) list}
```

Not Supported Nested Record Type Example in HOL

```
Datatype ‘file = Text string
               | Dir <| owner : string ;
                        files : (string # file) list |>‘
```

Filesystem Datatype - Mutual Recursion in HOL

```
Datatype ‘file = Text string
               | Dir directory
          ;
          directory = <| owner : string ;
                         files : (string # file) list |>‘
```

Datatype Package - No support for Co-Algebraic Types

- there is no support for co-algebraic types
- the Datatype package could be extended to do so
- other systems like Isabelle/HOL provide high-level methods for defining such types

Co-algebraic Type Example in SML — Lazy Lists

```
datatype 'a lazylist = Nil
                     | Cons of ('a * (unit -> 'a lazylist))
```

Datatype Package - Discussion

- Datatype package allows to define many useful datatypes
- however, there are many limitations
  - some types cannot be defined in HOL, e. g. empty types
  - some types are not supported, e. g. co-algebraic types
  - there are bugs (currently e. g. some trouble with certain mutually recursive definitions)
- biggest restrictions in practice (in my opinion and my line of work)
  - no support for co-algebraic datatypes
  - no nested record datatypes
- depending on datatype, different sets of useful lemmata are derived
- most important ones are added to TypeBase
  - tools like Induct_on, Cases_on use them
  - there is support for pattern matching

Total Functional Language (TFL) package

- TFL package implements support for terminating functional definitions
- Define defines functions from high-level descriptions
- there is support for pattern matching
- look and feel is like function definitions in SML
- based on **well-founded recursion** principle
- Define is the most common way for definitions in HOL

Well-Founded Relations

- a relation `R : 'a -> 'a -> bool` is called **well-founded**, iff there are no infinite descending chains

  ```
  wellfounded R = ~?f. !n. R (f (SUC n)) (f n)
  ```

- Example: `$< : num -> num -> bool` is well-founded
- if arguments of recursive calls are smaller according to well-founded relation, the recursion terminates
- this is the essence of termination proofs

Well-Founded Recursion

- a well-founded relation `R` can be used to define recursive functions
- this recursion principle is called `WFREC` in HOL
- idea of `WFREC`
  - if arguments get smaller according to `R`, perform recursive call
  - otherwise abort and return `ARB`
- `WFREC` always defines a function
- if all recursive calls indeed decrease according to `R`, the original recursive equations can be derived from the `WFREC` representation
- TFL uses this internally
- however, this is well-hidden from the user

Define - Initial Examples

Simple Definitions

```
> val DOUBLE_def = Define ‘DOUBLE n = n + n‘
val DOUBLE_def =
   |- !n. DOUBLE n = n + n:
   thm

> val MY_LENGTH_def = Define ‘(MY_LENGTH [] = 0) /\
                              (MY_LENGTH (x::xs) = SUC (MY_LENGTH xs))‘
val MY_LENGTH_def =
   |- (MY_LENGTH [] = 0) /\ !x xs. MY_LENGTH (x::xs) = SUC (MY_LENGTH xs):
   thm

> val MY_APPEND_def = Define ‘(MY_APPEND [] ys = ys) /\
                              (MY_APPEND (x::xs) ys = x :: (MY_APPEND xs ys))‘
val MY_APPEND_def =
   |- (!ys. MY_APPEND [] ys = ys) /\
      (!x xs ys. MY_APPEND (x::xs) ys = x::MY_APPEND xs ys):
   thm
```

`Define` discussion

- `Define` feels like a function definition in HOL
- it can be used to define "terminating" recursive functions
- `Define` is implemented by a large, non-trivial piece of SML code
- it uses many heuristics
- outcome of `Define` sometimes hard to predict
- the input descriptions are only hints
  - the produced function and the definitional theorem might be different
  - in simple examples, quantifiers added
  - pattern compilation takes place
  - earlier “conjuncts” have precedence

Define - More Examples

```
> val MY_HD_def = Define ‘MY_HD (x :: xs) = x‘
val MY_HD_def = |- !x xs. MY_HD (x::xs) = x : thm

> val IS_SORTED_def = Define ‘
     (IS_SORTED (x1 :: x2 :: xs) = ((x1 < x2) /\ (IS_SORTED (x2::xs)))) /\
     (IS_SORTED _ = T)‘
val IS_SORTED_def =
   |- (!xs x2 x1. IS_SORTED (x1::x2::xs) <=> x1 < x2 /\ IS_SORTED (x2::xs)) /\
      (IS_SORTED [] <=> T) /\ (!v. IS_SORTED [v] <=> T)

> val EVEN_def = Define ‘(EVEN 0 = T) /\ (ODD 0 = F) /\
                         (EVEN (SUC n) = ODD n) /\ (ODD (SUC n) = EVEN n)‘
val EVEN_def =
   |- (EVEN 0 <=> T) /\ (ODD 0 <=> F) /\ (!n. EVEN (SUC n) <=> ODD n) /\
      (!n. ODD (SUC n) <=> EVEN n) : thm

> val ZIP_def = Define ‘(ZIP (x::xs) (y::ys) = (x,y)::(ZIP xs ys)) /\
                        (ZIP _ _ = [])‘
val ZIP_def =
   |- (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys) /\
      (!v1. ZIP [] v1 = []) /\ (!v4 v3. ZIP (v3::v4) [] = []) : thm
```

Primitive Definitions

- Define introduces (if needed) the function using WFREC
- intended definition derived as a theorem
- the theorems are stored in current theory
- usually, one never needs to look at it

Examples

```
val IS_SORTED_primitive_def =
|- IS_SORTED =
   WFREC (@R. WF R /\ !x1 xs x2. R (x2::xs) (x1::x2::xs))
     (\IS_SORTED a.
        case a of
          [] => I T
        | [x1] => I T
        | x1::x2::xs => I (x1 < x2 /\ IS_SORTED (x2::xs)))

|- !R M. WF R ==> !x. WFREC R M x = M (RESTRICT (WFREC R M) R x) x
|- !f R x. RESTRICT f R x = (\y. if R y x then f y else ARB)
```

Induction Theorems

- Define automatically defines induction theorems
- these theorems are stored in current theory with suffix ind
- use DB.fetch "-" "something_ind" to retrieve them
- these induction theorems are useful to reason about corresponding recursive functions

Example

```
val IS_SORTED_ind = |- !P.
     ((!x1 x2 xs. P (x2::xs) ==> P (x1::x2::xs)) /\
      P [] /\
      (!v. P [v])) ==>
     !v. P v
```

Define failing

- Define might fail for various reasons to define a function
  - such a function cannot be defined in HOL
  - such a function can be defined, but not via the methods used by TFL
  - TFL can define such a function, but its heuristics are too weak and user guidance is required
  - there is a bug :-)
- **termination** is an important concept for Define
- it is easy to misunderstand termination in the context of HOL
- we need to understand what is meant by termination

Termination in HOL

- in SML it is natural to talk about termination of functions
- in the HOL logic there is no concept of execution
- thus, there is no concept of termination in HOL

3 characterisations of a function `f : num -> num`

- `|- !n. f n = 0`
- `|- (f 0 = 0) /\ !n. (f (SUC n) = f n)`
- `|- (f 0 = 0) /\ !n. (f n = f (SUC n))`

Is `f` terminating? All 3 theorems are equivalent.

Termination in HOL II

- it is useful to think in terms of termination
- the TFL package implements heuristics to define functions that would terminate in SML
- the TFL package uses well-founded recursion
- the required well-founded relation corresponds to a termination proof
- therefore, it is very natural to think of `Define` searching a termination proof
- important: this is the idea behind this function definition package, not a property of HOL

HOL is not limited to "terminating" functions

Termination in HOL III

- one can define "non-terminating" functions in HOL
- however, one cannot do so (easily) with `Define`

Definition of `WHILE` in HOL

```
|- !P g x. WHILE P g x = if P x then WHILE P g (g x) else x
```

Execution Order

There is no "execution order". One can easily define a complicated constant function:

```
(myk : num -> num) (n:num) = (let x = myk (n+1) in 0)
```

Unsound Definitions

A function `f : num -> num` with the following property cannot be defined in HOL unless HOL has an inconsistancy:

```
!n. f n = ((f n) + 1)
```

Such a function would allow to prove `0 = 1`.

Manual Termination Proofs I

- TFL uses various heuristics to find a well-founded relation
- however, these heuristics may not be strong enough
- in such cases the user can provide a well-founded relation manually
- the most common well-founded relations are **measures**
- measures map values to natural numbers and use the less relation
  `|- !(f:'a -> num) x y. measure f x y <=> (f x < f y)`
- all measures are well-founded: `|- !f. WF (measure f)`
- moreover, existing well-founded relations can be combined
  - lexicographic order `LEX`
  - list lexicographic order `LLEX`
  - . . .

Manual Termination Proofs II

- if `Define` fails to find a termination proof, `Hol_defn` can be used
- `Hol_defn` defers termination proofs
- it derives termination conditions and sets up the function definitions
- all results are packaged as a value of type `defn`
- after calling `Hol_defn` the defined function(s) can be used
- however, the intended definition theorem has not been derived yet
- to derive it, one needs to
  - provide a well-founded relation
  - show that termination conditions respect that relation
- `Defn.tprove` and `Defn.tgoal` are intended for this
- proofs usually start by providing relation via tactic `WF_REL_TAC`

Manual Termination Proof Example 1

```
> val qsort_defn = Hol_defn "qsort" ‘
  (qsort ord [] = []) /\
  (qsort ord (x::rst) =
     (qsort ord (FILTER ($~ o ord x) rst)) ++
     [x] ++
     (qsort ord (FILTER (ord x) rst)))‘

val qsort_defn = HOL function definition (recursive)

Equation(s) :
 [...] |- qsort ord [] = []
 [...] |- qsort ord (x::rst) =
           qsort ord (FILTER ($~ o ord x) rst) ++ [x] ++
           qsort ord (FILTER (ord x) rst)

Induction : ...

Termination conditions :
  0. !rst x ord. R (ord,FILTER (ord x) rst) (ord,x::rst)
  1. !rst x ord. R (ord,FILTER ($~ o ord x) rst) (ord,x::rst)
  2. WF R
```

Manual Termination Proof Example 2

```
> Defn.tgoal qsort_defn

Initial goal:

?R.
  WF R /\
  (!rst x ord. R (ord,FILTER (ord x)      rst) (ord,x::rst)) /\
  (!rst x ord. R (ord,FILTER ($~ o ord x) rst) (ord,x::rst))
```

Manual Termination Proof Example 2

```
> Defn.tgoal qsort_defn

Initial goal:

?R.
  WF R /\
  (!rst x ord. R (ord,FILTER (ord x)      rst) (ord,x::rst)) /\
  (!rst x ord. R (ord,FILTER ($~ o ord x) rst) (ord,x::rst))


> e (WF_REL_TAC ‘measure (\(_, l). LENGTH l)‘)

1 subgoal :

(!rst x ord. LENGTH (FILTER (ord x) rst) < LENGTH (x::rst)) /\
(!rst x ord. LENGTH (FILTER (\x’. ~ord x x’) rst) < LENGTH (x::rst))

> ...
```

Manual Termination Proof Example 3

```
> val (qsort_def, qsort_ind) =
  Defn.tprove (qsort_defn,
    WF_REL_TAC `measure (\(_, l). LENGTH l)` >> ...)

val qsort_def =
|- (qsort ord [] = []) /\
   (qsort ord (x::rst) =
    qsort ord (FILTER ($~ o ord x) rst) ++ [x] ++
    qsort ord (FILTER (ord x) rst))

val qsort_ind =
|- !P. (!ord. P ord []) /\
       (!ord x rst.
          P ord (FILTER (ord x) rst) /\
          P ord (FILTER ($~ o ord x) rst) ==>
          P ord (x::rst)) ==>
       !v v1. P v v1
```

Part XI

Good Definitions

Importance of Good Definitions

- using *good* definitions is very important
  - good definitions are vital for **clarity**
  - **proofs** depend a lot on the form of definitions
- unluckily, it is hard to state what a good definition is
- even harder to come up with good definitions
- let's look at it a bit closer anyhow

Importance of Good Definitions — Clarity I

- HOL guarantees that theorems do indeed hold
- However, does the theorem mean what you think it does?
- you can separate your development in
  - main theorems you care for
  - auxiliary stuff used to derive your main theorems
- it is essential to understand your main theorems

Importance of Good Definitions — Clarity II

Guarded by HOL

- proofs checked
- internal, technical definitions
- technical lemmata
- proof tools

Manual review needed for

- meaning of main theorems
- meaning of definitions used by main theorems
- meaning of types used by main theorems

Importance of Good Definitions — Clarity III

- it is essential to understand your main theorems
  - you need to understand all the definitions directly used
  - you need to understand the indirectly used ones as well
  - you need to convince others that you express the intended statement
  - therefore, it is vital to use very simple, clear definitions
- defining concepts is often the main development task
- checking resulting model against real artefact is vital
  - testing via e. g. `EVAL`
  - formal sanity
  - conformance testing
- wrong models are main source of error when using HOL
- proofs, auxiliary lemmata and auxiliary definitions
  - can be as technical and complicated as you like
  - correctness is guaranteed by HOL
  - reviewers don't need to care

Importance of Good Definitions — Proofs

- good definitions can shorten proofs significantly
- they improve maintainability
- they can improve automation drastically
- unluckily for proofs definitions often need to be technical
- this contradicts clarity aims

How to come up with good definitions

- unluckily, it is hard to state what a good definition is
- it is even harder to come up with them
  - there are often many competing interests
  - a lot of experience and detailed tool knowledge is needed
  - much depends on personal style and taste
- general advice: use more than one definition
  - in HOL you can derive equivalent definitions as theorems
  - define a concept as clearly and easily as possible
  - derive equivalent definitions for various purposes
    - one very close to your favourite textbook
    - one nice for certain types of proofs
    - another one good for evaluation
    - . . .
- lessons from functional programming apply

Good Definitions in Functional Programming

Objectives

- clarity (readability, maintainability)
- performance (runtime speed, memory usage, ...)

General Advice

- use the powerful type-system
- use many small function definitions
- encode invariants in types and function signatures

Good Definitions – no number encodings

- many programmers familiar with C encode everything as a number
- enumeration types are very cheap in SML and HOL
- use them instead

Example Enumeration Types

In C the result of an order comparison is an integer with 3 equivalence classes: 0, negative and positive integers. In SML and HOL, it is better to use a variant type.

```
val _ = Datatype ‘ordering = LESS | EQUAL | GREATER‘;

val compare_def = Define ‘
   (compare LESS    lt eq gt = lt)
/\ (compare EQUAL   lt eq gt = eq)
/\ (compare GREATER lt eq gt = gt) ‘;

val list_compare_def = Define ‘
   (list_compare cmp [] [] = EQUAL) /\ (list_compare cmp [] l2 = LESS)
/\ (list_compare cmp l1 [] = GREATER)
/\ (list_compare cmp (x::l1) (y::l2) = compare (cmp (x:’a) y)
     (* x<y *) LESS
     (* x=y *) (list_compare cmp l1 l2)
     (* x>y *) GREATER) ‘;
```

Good Definitions — Isomorphic Types

- the type-checker is your friend
  - it helps you find errors
  - code becomes more robust
  - using good types is a great way of writing self-documenting code
- therefore, use many types
- even use types isomorphic to existing ones

Virtual and Physical Memory Addresses

Virtual and physical addresses might in a development both be numbers. It is still nice to use separate types to avoid mixing them up.

```
val _ = Datatype ‘vaddr = VAddr num‘;
val _ = Datatype ‘paddr = PAddr num‘;

val virt_to_phys_addr_def = Define ‘
  virt_to_phys_addr (VAddr a) = PAddr( translation of  a )‘;
```

Good Definitions — Record Types I

- often people use tuples where records would be more appropriate
- using large tuples quickly becomes awkward
  - it is easy to mix up order of tuple entries
    - often types coincide, so type-checker does not help
  - no good error messages for tuples
    - hard to decipher type mismatch messages for long product types
    - hard to figure out which entry is missing at which position
    - non-local error messages
    - variable in last entry can hide missing entries
- records sometimes require slightly more proof effort
- however, records have many benefits

Good Definitions — Record Types II

- using records
  - introduces field names
  - provides automatically defined accessor and update functions
  - leads to better type-checking error messages
- records improve readability
  - accessors and update functions lead to shorter code
  - field names act as documentation
- records improve maintainability
  - improved error messages
  - much easier to add extra fields

Good Definitions — Encoding Invariants

- try to encode as many invariants as possible in the types
- this allows the type-checker to ensure them for you
- you don't have to check them manually any more
- your code becomes more robust and clearer

Network Connections (Example by Yaron Minsky from Jane Street)

Consider the following datatype for network connections. It has many implicit invariants.

```
datatype connection_state = Connected | Disconnected | Connecting;

type connection_info = {
  state              : connection_state,
  server             : inet_address,
  last_ping_time     : time option,
  last_ping_id       : int option,
  session_id         : string option,
  when_initiated     : time option,
  when_disconnected  : time option
}
```

Good Definitions — Encoding Invariants II

Network Connections (Example by Yaron Minsky from Jane Street) II

The following definition of connection_info makes the invariants explicit:

```
type connected    = { last_ping         : (time * int) option,
                      session_id        : string };
type disconnected = { when_disconnected : time };
type connecting   = { when_initiated    : time };

datatype connection_state =
  Connected of connected
| Disconnected of disconneted
| Connecting of connecting;

type connection_info = {
  state  : connection_state,
  server : inet_address
}
```

Good Definitions in HOL

Objectives

- clarity (readability)
- good for proofs
- performance (good for automation, easily evaluatable, ...)

General Advice

- same advice as for functional programming applies
- use even smaller definitions
  - introduce auxiliary definitions for important function parts
  - use extra definitions for important constants
  - ...
- tiny definitions
  - allow keeping proof state small by unfolding only needed ones
  - allow many small lemmata
  - improve maintainability

Good Definitions in HOL II

Technical Issues

- write definition such that they work well with HOL's tools
- this requires you to know HOL well
- a lot of experience is required
- general advice
    - avoid explicit case-expressions
    - prefer curried functions

Example

```
val ZIP_GOOD_def = Define `(ZIP (x::xs) (y::ys) = (x,y)::(ZIP xs ys)) /\
                           (ZIP _ _ = [])`

val ZIP_BAD1_def = Define `ZIP xs ys = case (xs, ys) of
                                          (x::xs, y::ys) => (x,y)::(ZIP xs ys)
                                        | (_, _) => []`

val ZIP_BAD2_def = Define `(ZIP (x::xs, y::ys) = (x,y)::(ZIP (xs, ys))) /\
                           (ZIP _ = [])`
```

Good Definitions in HOL III

Multiple Equivalent Definitions

- satisfy competing requirements by having multiple equivalent definitions
- derive them as theorems
- initial definition should be as clear as possible
  - clarity allows simpler reviews
  - simplicity reduces the likelihood of errors

Example - ALL_DISTINCT

```
|- (ALL_DISTINCT [] <=> T) /\
   (!h t. ALL_DISTINCT (h::t) <=> ~MEM h t /\ ALL_DISTINCT t)

|- !l. ALL_DISTINCT l <=>
       (!x. MEM x l ==> (FILTER ($= x) l = [x]))

|- !ls. ALL_DISTINCT ls <=> (CARD (set ls) = LENGTH ls):
```

Formal Sanity

Formal Sanity

- to ensure correctness test your definitions via e. g. `EVAL`
- in HOL testing means symbolic evaluation, i. e. proving lemmata
- **formally proving sanity check lemmata** is very beneficial
  - they should express core properties of your definition
  - thereby they check your intuition against your actual definitions
  - these lemmata are often useful for following proofs
  - using them improves robustness and maintainability of your development
- I highly recommend using formal sanity checks

Formal Sanity Example I

```
> val ALL_DISTINCT = Define ‘
    (ALL_DISTINCT [] = T) /\
    (ALL_DISTINCT (h::t) = ~MEM h t /\ ALL_DISTINCT t)‘;
```

Example Sanity Check Lemmata

```
|- ALL_DISTINCT []

|- !x xs. ALL_DISTINCT (x::xs) <=> ~MEM x xs /\ ALL_DISTINCT xs

|- !x. ALL_DISTINCT [x]

|- !x xs. ~(ALL_DISTINCT (x::x::xs))

|- !l. ALL_DISTINCT (REVERSE l) <=> ALL_DISTINCT l

|- !x l. ALL_DISTINCT (SNOC x l) <=> ~MEM x l /\ ALL_DISTINCT l

|- !l1 l2. ALL_DISTINCT (l1 ++ l2) <=>
           ALL_DISTINCT l1 /\ ALL_DISTINCT l2 /\ !e. MEM e l1 ==> ~MEM e l2
```

Formal Sanity Example II 1

```
> val ZIP_def = Define `
    (ZIP [] ys = []) /\ (ZIP xs [] = []) /\
    (ZIP (x::xs) (y::ys) = (x, y)::(ZIP xs ys))`

val ZIP_def =
 |- (!ys. ZIP [] ys = []) /\ (!v3 v2. ZIP (v2::v3) [] = []) /\
    (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys)
```

- above definition of ZIP looks straightforward
- small changes cause heuristics to produce different theorems
- use formal sanity lemmata to compensate

```
> val ZIP_def = Define `
    (ZIP xs [] = []) /\ (ZIP [] ys = []) /\
    (ZIP (x::xs) (y::ys) = (x, y)::(ZIP xs ys))`

val ZIP_def =
   |- (!xs. ZIP xs [] = []) /\ (!v3 v2. ZIP [] (v2::v3) = []) /\
      (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys0
```

Formal Sanity Example II 2

```
val ZIP_def =
 |- (!ys. ZIP [] ys = []) /\ (!v3 v2. ZIP (v2::v3) [] = []) /\
    (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys)
```

Example Formal Sanity Lemmata

```
|- (!xs. ZIP xs [] = []) /\ (!ys. ZIP [] ys = []) /\
   (!y ys x xs. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys)

|- !xs ys. LENGTH (ZIP xs ys) = MIN (LENGTH xs) (LENGTH ys)

|- !x y xs ys. MEM (x, y) (ZIP xs ys) ==> (MEM x xs /\ MEM y ys)

|- !xs1 xs2 ys1 ys2. LENGTH xs1 = LENGTH ys1 ==>
                     (ZIP (xs1++xs2) (ys1++ys2) = (ZIP xs1 ys1 ++ ZIP xs2 ys2))

...
```

- in your proofs use sanity lemmata, not original definition
- this makes your development robust against
  - small changes to the definition required later
  - changes to `Define` and its heuristics
  - bugs in function definition package

Part XII

Deep and Shallow Embeddings

Deep and Shallow Embeddings

- often one models some kind of formal language
- important design decision: use **deep** or **shallow** embedding
- in a nutshell:
  - shallow embeddings just model semantics
  - deep embeddings model syntax as well
- a shallow embedding directly uses the HOL logic
- a deep embedding
  - defines a datatype for the syntax of the language
  - provides a function to map this syntax to a semantic

Example: Embedding of Propositional Logic I

- propositional logic is a subset of HOL
- a shallow embedding is therefore trivial

```
val sh_true_def    = Define ‘sh_true = T‘;
val sh_var_def     = Define ‘sh_var (v:bool) = v‘;
val sh_not_def     = Define ‘sh_not b = ~b‘;
val sh_and_def     = Define ‘sh_and b1 b2 = (b1 /\ b2)‘;
val sh_or_def      = Define ‘sh_or b1 b2 = (b1 \/ b2)‘;
val sh_implies_def = Define ‘sh_implies b1 b2 = (b1 ==> b2)‘;
```

Example: Embedding of Propositional Logic II

- we can also define a datatype for propositional logic
- this leads to a deep embedding

```
val _ = Datatype ‘bvar = BVar num‘
val _ = Datatype ‘prop = d_true | d_var bvar | d_not prop
                       | d_and prop prop | d_or prop prop
                       | d_implies prop prop‘;

val _ = Datatype ‘var_assignment = BAssign (bvar -> bool)‘
val VAR_VALUE_def = Define ‘VAR_VALUE (BAssign a) v = (a v)‘

val PROP_SEM_def = Define ‘
  (PROP_SEM a d_true = T) /\
  (PROP_SEM a (d_var v) = VAR_VALUE a v) /\
  (PROP_SEM a (d_not p) = ~(PROP_SEM a p)) /\
  (PROP_SEM a (d_and p1 p2) = (PROP_SEM a p1 /\ PROP_SEM a p2)) /\
  (PROP_SEM a (d_or p1 p2) = (PROP_SEM a p1 \/ PROP_SEM a p2)) /\
  (PROP_SEM a (d_implies p1 p2) = (PROP_SEM a p1 ==> PROP_SEM a p2))‘
```

Shallow vs. Deep Embeddings

Shallow

- quick and easy to build
- extensions are simple

Deep

- can reason about syntax
- allows verified implementations
- sometimes tricky to define
  - e. g. bound variables

Important Questions for Deciding

- Do I need to reason about syntax?
- Do I have hard to define syntax like bound variables?
- How much time do I have?

Example: Embedding of Propositional Logic III

- with deep embedding one can easily formalise syntactic properties like
  - Which variables does a propositional formula contain?
  - Is a formula in negation-normal-form (NNF)?
- with shallow embeddings
  - syntactic concepts can't be defined in HOL
  - however, they can be defined in SML
  - no proofs about them possible

```
val _ = Define ‘
  (IS_NNF (d_not d_true) = T) /\ (IS_NNF (d_not (d_var v)) = T) /\
  (IS_NNF (d_not _) = F) /\

  (IS_NNF d_true = T) /\ (IS_NNF (d_var v) = T) /\
  (IS_NNF (d_and p1 p2) = (IS_NNF p1 /\ IS_NNF p2)) /\
  (IS_NNF (d_or p1 p2) = (IS_NNF p1 /\ IS_NNF p2)) /\
  (IS_NNF (d_implies p1 p2) = (IS_NNF p1 /\ IS_NNF p2))‘
```

Verified vs. Verifying Program

Verified Programs

- are formalised in HOL
- their properties have been proven once and for all
- all runs have proven properties
- are usually less sophisticated, since they need verification
- is what one wants ideally
- often require deep embedding

Verifying Programs

- are written in meta-language
- they produce a separate proof for each run
- only certain that current run has properties
- allow more flexibility, e. g. fancy heuristics
- good pragmatic solution
- shallow embedding fine

Summary Deep vs. Shallow Embeddings

- deep embeddings require more work
- they however allow reasoning about syntax
  - induction and case-splits possible
  - a semantic subset can be carved out syntactically
- syntax sometimes hard to define for deep embeddings
- combinatations of deep and shallow embeddings common
  - certain parts are deeply embedded
  - others are embedded shallowly

Part XIII

Rewriting

Rewriting in HOL

- simplification via rewriting was already a strength of Edinburgh LCF
- it was further improved for Cambridge LCF
- HOL inherited this powerful rewriter
- equational reasoning is still the main workhorse
- there are many different equational reasoning tools in HOL
  - `Rewrite` library
    inherited from Cambridge LCF
    you have seen it in the form of `REWRITE_TAC`
  - `computeLib` — fast evaluation
    build for speed, optimised for ground terms
    seen in the form of `EVAL`
  - `simpLib` — Simplification
    sophisticated rewrite engine, HOL's main workhorse
    not discussed in this lecture, yet
  - ...

Semantic Foundations

- we have seen primitive inference rules for equality before

$$\frac{\Gamma \vdash s = t \qquad \Delta \vdash u = v \qquad \textit{types fit}}{\Gamma \cup \Delta \vdash s(u) = t(v)}\ \mathrm{COMB} \qquad \frac{\Gamma \vdash s = t \qquad x \textit{ not free in } \Gamma}{\Gamma \vdash \lambda x.\ s = \lambda x.\ t}\ \mathrm{ABS}$$

$$\frac{\Gamma \vdash s = t \qquad \Delta \vdash t = u}{\Gamma \cup \Delta \vdash s = u}\ \mathrm{TRANS} \qquad \frac{}{\vdash t = t}\ \mathrm{REFL}$$

- these rules allow us to replace any subterm with an equal one
- this is the core of rewriting

Conversions

- in HOL, equality reasoning is implemented by **conversions**
- a conversion is a SML function of type `term -> thm`
- given a term `t`, a conversion
  - produces a theorem of the form `|- t = t'`
  - raises an `UNCHANGED` exception or
  - fails, i. e. raises an `HOL_ERR` exception

Example

```
> BETA_CONV ``(\x. SUC x) y``
val it = |- (\x. SUC x) y = SUC y

> BETA_CONV ``SUC y``
Exception-HOL_ERR ... raised

> REPEATC BETA_CONV ``SUC y``
Exception- UNCHANGED raised
```

Conversionals

- similar to tactics and tacticals there are **conversionals** for conversions
- conversionals allow building conversions from simpler ones
- there are many of them
  - `THENC`
  - `ORELSEC`
  - `REPEATC`
  - `TRY_CONV`
  - `RAND_CONV`
  - `RATOR_CONV`
  - `ABS_CONV`
  - ...

Depth Conversionals

- for rewriting depth-conversionals are important
- a depth-conversional applies a conversion to all subterms
- there are many different ones
  - `ONCE_DEPTH_CONV c` — top down, applies `c` once at highest possible positions in distinct subterms
  - `TOP_SWEEP_CONV c` — top down, like `ONCE_DEPTH_CONV`, but continues processing rewritten terms
  - `TOP_DEPTH_CONV c` — top down, like `TOP_SWEEP_CONV`, but try top-level again after change
  - `DEPTH_CONV c` — bottom up, recurse over subterms, then apply `c` repeatedly at top-level
  - `REDEPTH_CONV c` — bottom up, like `DEPTH_CONV`, but revisits subterms

REWR_CONV

- it remains to rewrite terms at top-level
- this is achieved by `REWR_CONV`
- given a term `t` and a theorem `|- t1 = t2`, `REWR_CONV t thm`
  - searches an instantiation of term and type variables such that `t1` becomes α-equivalent to `t`
  - fails, if no instantiation is found
  - otherwise, instantiate the theorem and get `|- t1’ = t2’`
  - return theorem `|- t = t2’`

Example

term `LENGTH [1;2;3]`, theorem `|- LENGTH ((x:’a)::xs) = SUC (LENGTH xs)`
found type instantiation: `[‘‘:’a‘‘ |-> ‘‘:num‘‘]`
found term instantiation: `[‘‘x:num‘‘ |-> ‘‘1‘‘; ‘‘xs‘‘ |-> ‘‘[2;3]‘‘]`
returned theorem: `|- LENGTH [1;2;3] = SUC (LENGTH [2;3])`

- the tricky part is finding the instantiation
- this problem is called the (term) **matching** problem

Term Matching

- given term `t_org` and a term `t_goal` try to find
  - type substitution `ty_s`
  - term substitution `tm_s`
- such that `subst tm_s (inst ty_s t_org)` $\overset{\alpha}{\equiv}$ `t_goal`
- this can be easily implemented by a recursive search

t_org	t_goal	**action**
`t1_org t2_org`	`t1_goal t2_goal`	recurse
`t1_org t2_org`	otherwise	fail
`\x. t_org x`	`\y. t_goal y`	match types of `x`, `y` and recurse
`\x. t_org x`	otherwise	fail
`const`	same const	match types
`const`	otherwise	fail
`var`	anything	try to bind `var`, take care of existing bindings

Examples Term Matching

t_org	t_goal	**substs**
LENGTH ((x:'a)::xs)	LENGTH [1;2;3]	'a → num, x → 1, xs → [2;3]
[]:'a list	[]:'b list	'a → 'b
0	0	empty substitution
b /\ T	(P (x:'a) ==> Q) /\ T	b → P x ==> Q
b /\ b	P x /\ P x	b → P x
b /\ b	P x /\ P y	fail
!x:num. P x /\ Q x	!y:num. P' y /\ Q' y	P → P', Q → Q'
!x:num. P x /\ Q x	!y. (2 = y) /\ Q' y	P → ($= 2), Q → Q'
!x:num. P x /\ Q x	!y. (y = 2) /\ Q' y	fail

- it is often very annoying that the last match fails
- it prevents us for example rewriting !y. (2 = y) /\ Q y to (!y. (2=y)) /\ (!y. Q y)
- Can we do better? Yes, with higher order (term) matching.

Higher Order Term Matching

- term matching searches for substitutions such that `t_org` becomes α-equivalent to `t_goal`
- **higher order term matching** searches for substitutions such that `t_org` becomes `t_subst` such that the $\beta\eta$-normalform of `t_subst` is α-equivalent equivalent to $\beta\eta$-normalform of `t_goal`, i. e.
  higher order term matching is aware of the semantics of λ

β-reduction	$(\lambda x.\ f)\ y = f[y/x]$
η-conversion	$(\lambda x.\ f\ x) = f$ where x is not free in f

- the HOL implementation expects `t_org` to be a **higher-order pattern**
  - `t_org` is β-reduced
  - if `X` is a variable that should be instantiated, then all arguments should be distinct variables
- for other forms of `t_org`, HOL's implementation might fail
- higher order matching is used by `HO_REWR_CONV`

Examples Higher Order Term Matching

`t_org`	`t_goal`	**substs**
`!x:num. P x /\ Q x`	`!y. (y = 2) /\ Q' y`	`P` → `(\y. y = 2)`, `Q` → `Q'`
`!x. P x /\ Q x`	`!x. P x /\ Q x /\ Z x`	`Q` → `\x. Q x /\ Z x`
`!x. P x /\ Q`	`!x. P x /\ Q x`	fails
`!x. P (x, x)`	`!x. Q x`	fails
`!x. P (x, x)`	`!x. FST (x,x) = SND (x,x)`	`P` → `\xx. FST xx = SND xx`

Don't worry, it might look complicated, but in practice it is easy to get a feeling for higher order matching.

Rewrite Library

- the rewrite library combines REWR_CONV with depth conversions
- there are many different conversions, rules and tactics
- at they core, they all work very similarly
  - given a list of theorems, a set of rewrite theorems is derived
    - split conjunctions
    - remove outermost universal quantification
    - introduce equations by adding = T (or = F) if needed
  - REWR_CONV is applied to all the resulting rewrite theorems
  - a depth-conversion is used with resulting conversion
- for performance reasons an efficient indexing structure is used
- by default implicit rewrites are added

Rewrite Library II

- REWRITE_CONV
- REWRITE_RULE
- REWRITE_TAC
- ASM_REWRITE_TAC
- ONCE_REWRITE_TAC
- PURE_REWRITE_TAC
- PURE_ONCE_REWRITE_TAC
- ...

Ho_Rewrite Library

- similar to Rewrite lib, but uses higher order matching
- internally uses HO_REWR_CONV
- similar conversions, rules and tactics as Rewrite lib
  - Ho_Rewrite.REWRITE_CONV
  - Ho_Rewrite.REWRITE_RULE
  - Ho_Rewrite.REWRITE_TAC
  - Ho_Rewrite.ASM_REWRITE_TAC
  - Ho_Rewrite.ONCE_REWRITE_TAC
  - Ho_Rewrite.PURE_REWRITE_TAC
  - Ho_Rewrite.PURE_ONCE_REWRITE_TAC
  - ...

Examples Rewrite and Ho_Rewrite Library

```
> REWRITE_CONV [LENGTH] ``LENGTH [1;2]``
val it = |- LENGTH [1; 2] = SUC (SUC 0)

> ONCE_REWRITE_CONV [LENGTH] ``LENGTH [1;2]``
val it = |- LENGTH [1; 2] = SUC (LENGTH [2])

> REWRITE_CONV [] ``A /\ A /\ ~A``
Exception- UNCHANGED raised

> PURE_REWRITE_CONV [NOT_AND] ``A /\ A /\ ~A``
val it = |- A /\ A /\ ~A <=> A /\ F

> REWRITE_CONV [NOT_AND] ``A /\ A /\ ~A``
val it = |- A /\ A /\ ~A <=> F

> REWRITE_CONV [FORALL_AND_THM] ``!x. P x /\ Q x /\ R x``
Exception- UNCHANGED raised

> Ho_Rewrite.REWRITE_CONV [FORALL_AND_THM] ``!x. P x /\ Q x /\ R x``
val it = |- !x. P x /\ Q x /\ R x <=> (!x. P x) /\ (!x. Q x) /\ (!x. R x)
```

Summary Rewrite and Ho_Rewrite Library

- the Rewrite and Ho_Rewrite library provide powerful infrastructure for term rewriting
- thanks to clever implementations they are reasonably efficient
- basics are easily explained
- however, efficient usage needs some experience

Term Rewriting Systems

- to use rewriting efficiently, one needs to understand about term rewriting systems
- this is a large topic
- one can easily give whole course just about term rewriting systems
- however, in practise you quickly get a feeling
- important points in practise
  - ensure termination of your rewrites
  - make sure they work nicely together

Term Rewriting Systems — Termination

Theory

- choose well-founded order $\prec$
- for each rewrite theorem `|- t1 = t2` ensure `t2` $\prec$ `t1`

Practice

- informally define for yourself what **simpler** means
- ensure each rewrite makes terms simpler
- good heuristics
  - subterms are simpler than whole term
  - use an order on functions

Termination — Subterm examples

- a proper subterm is always simpler
  - `!l. APPEND [] l = l`
  - `!n. n + 0 = n`
  - `!l. REVERSE (REVERSE l) = l`
  - `!t1 t2. if T then t1 else t2 <=> t1`
  - `!n. n * 0 = 0`
- the right hand side should not use extra vars, throwing parts away is usually simpler
  - `!x xs. (SNOC x xs = []) = F`
  - `!x xs. LENGTH (x::xs) = SUC (LENGTH xs)`
  - `!n x xs. DROP (SUC n) (x::xs) = DROP n xs`

Termination — use simpler terms

- it is useful to consider some functions simple and other complicated
- replace complicated ones with simple ones
- never do it in the opposite direction
- clear examples
  - `|- !m n. MEM m (COUNT_LIST n) <=> (m < n)`
  - `|- !ls n. (DROP n ls = []) <=> (n >= LENGTH ls)`
- unclear example
  - `|- !L. REVERSE L = REV L []`

Termination — Normalforms

- some equations can be used in both directions
- one should decide on one direction
- this implicitly defined a **normalform** one wants terms to be in
- examples
  - `|- !f l. MAP f (REVERSE l) = REVERSE (MAP f l)`
  - `|- !l1 l2 l3. l1 ++ (l2 ++ l3) = l1 ++ l2 ++ l3`

Termination — Problematic rewrite rules

- some equations immediately lead to non-termination, e. g.
  - `|- !m n. m + n = n + m`
  - `|- !m. m = m + 0`
- slightly more subtle are rules like
  - `|- !n. fact n = if (n = 0) then 1 else n * fact(n-1)`
- often combination of multiple rules leads to non-termination this is especially problematic when adding to predefined set of rewrites
  - `|- !m n p. m + (n + p) = (m + n) + p` and
    `|- !m n p. (m + n) + p = m + (n + p)`

Rewrites working together

- rewrite rules should not complete with each other
- if a term `ta` can be rewritten to `ta1` and `ta2` applying different rewrite rules, then the `ta1` and `ta2` should be further rewritten to a common `tb`
- this can often be achieved by adding extra rewrite rules

Example

Assume we have the rewrite rules `|- DOUBLE n = n + n` and `|- EVEN (DOUBLE n) = T`.
With these the term `EVEN (DOUBLE 2)` can be rewritten to

- `T` or
- `EVEN (2 + 2)`.

To avoid a hard to predict result, `EVEN (2+2)` should be rewritten to `T`.
Adding an extra rewrite rule `|- EVEN (n + n) = T` achieves this.

Rewrites working together II

- to design rewrite systems that work well, normalforms are vital
- a term is in **normalform**, if it cannot be rewritten any further
- one should have a clear idea what the normalform of common terms looks like
- all rules should work together to establish this normalform
- the right-hand-side of each rule should be in normalform
- the left-hand-side should not be simplifiable by any other rule
- the order in which rules are applied should not influence the final result

computeLib

- computeLib is the library behind EVAL
- it is a rewriting library designed for evaluating ground terms (i. e. terms without variables) efficiently
- it uses a call-by-value strategy similar to SML's
- it uses first order term matching
- it performs β reduction in addition to rewrites

compset

- computeLib uses compsets to store its rewrites
- a compset stores
  - rewrite rules
  - extra conversions
- the extra conversions are guarded by a term pattern for efficiency
- users can define their own compsets
- however, computeLib maintains one special compset called the_compset
- the_compset is used by EVAL

EVAL

- EVAL uses the_compset
- tools like the Datatype of TFL automatically extend the_compset
- this way, EVAL knows about (nearly) all types and functions
- one can extended the_compset manually as well
- rewrites exported by Define are good for ground terms but may lead to non-termination for non-ground terms
- zDefine prevents TFL from automatically extending the_compset

simpLib

- simpLib is a sophisticated rewrite engine
- it is HOL's main workhorse
- it provides
  - higher order rewriting
  - usage of context information
  - conditional rewriting
  - arbitrary conversions
  - support for decision procedures
  - simple heuristics to avoid non-termination
  - fancier preprocessing of rewrite theorems
  - ...
- it is very powerful, but compared to Rewrite lib sometimes slow

Basic Usage I

- `simpLib` uses **simpsets**
- simpsets are special datatypes storing
  - rewrite rules
  - conversions
  - decision procedures
  - congruence rules
  - ...
- in addition there are simpset-fragments
- simpset-fragments contain similar information as simpsets
- fragments can be added to and removed from simpsets
- common usage: basic simpset combined with one or more simpset-fragments, e. g.
  - `list_ss ++ pairSimps.gen_beta_ss`
  - `std_ss ++ QI_ss`
  - ...

Basic Usage II

- a call to the simplifier takes as arguments
  - a simpset
  - a list of rewrite theorems
- common high-level entry points are
  - `SIMP_CONV ss thmL` — conversion
  - `SIMP_RULE ss thmL` — rule
  - `SIMP_TAC ss thmL` — tactic without considering assumptions
  - `ASM_SIMP_TAC ss thmL` — tactic using assumptions to simplify goal
  - `FULL_SIMP_TAC ss thmL` — tactic simplifying assumptions with each other and goal with assumptions
  - `REV_FULL_SIMP_TAC ss thmL` — similar to `FULL_SIMP_TAC` but with reversed order of assumptions
- there are many derived tools not discussed here

Basic Simplifier Examples

```
> SIMP_CONV bool_ss [LENGTH] ``LENGTH [1;2]``
val it = |- LENGTH [1; 2] = SUC (SUC 0)

> SIMP_CONV std_ss [LENGTH] ``LENGTH [1;2]``
val it = |- LENGTH [1; 2] = 2

> SIMP_CONV list_ss [] ``LENGTH [1;2]``
val it = |- LENGTH [1; 2] = 2
```

FULL_SIMP_TAC Example

Current GoalStack

```
P (SUC (SUC x0)) (SUC (SUC y0))
------------------------------------
  0.  SUC y1 = y2
  1.  x1 = SUC x0
  2.  y1 = SUC y0
  3.  SUC x1 = x2
```

Action

```
FULL_SIMP_TAC std_ss []
```

Resulting GoalStack

```
P (SUC (SUC x0)) y2
------------------------------------
  0.  SUC (SUC y0) = y2
  1.  x1 = SUC x0
  2.  y1 = SUC y0
  3.  SUC x1 = x2
```

REV_FULL_SIMP_TAC Example

Current GoalStack

```
P (SUC (SUC x0)) y2
------------------------------------
  0.  SUC (SUC y0) = y2
  1.  x1 = SUC x0
  2.  y1 = SUC y0
  3.  SUC x1 = x2
```

Action

```
REV_FULL_SIMP_TAC std_ss []
```

Resulting GoalStack

```
P x2 y2
------------------------------------
  0.  SUC (SUC y0) = y2
  1.  x1 = SUC x0
  2.  y1 = SUC y0
  3.  SUC (SUC x0) = x2
```

Common simpsets

- pure_ss — empty simpset
- bool_ss — basic simpset
- std_ss — standard simpset
- arith_ss — arithmetic simpset
- list_ss — list simpset
- real_ss — real simpset

Common simpset-fragments

- many theories and libraries provide their own simpset-fragments
- `PRED_SET_ss` — simplify sets
- `STRING_ss` — simplify strings
- `QI_ss` — extra quantifier instantiations
- `gen_beta_ss` — β reduction for pairs
- `ETA_ss` — η conversion
- `EQUIV_EXTRACT_ss` — extract common part of equivalence
- `CONJ_ss` — use conjunctions for context
- ...

Build-In Conversions and Decision Procedures

- in contrast to `Rewrite` lib the simplifier can run arbitrary conversions
- most useful is probably β reduction
- `std_ss` has support for basic arithmetic and numerals
- it also has simple, syntactic conversions for instantiating quantifiers
  - `!x.  ...  /\ (x = c) /\ ...  ==> ...`
  - `!x.  ...  \/ ~(x = c) \/ ...`
  - `?x.  ...  /\ (x = c) /\ ...`
- besides very useful conversions, there are decision procedures as well
- the most frequently used one is probably the arithmetic decision procedure you already know from `DECIDE`

Examples I

```
> SIMP_CONV std_ss [] ‘‘(\x. x + 2) 5‘‘
val it = |- (\x. x + 2) 5 = 7

> SIMP_CONV std_ss [] ‘‘!x. Q x /\ (x = 7) ==> P x‘‘
val it = |- (!x. Q x /\ (x = 7) ==> P x) <=> (Q 7 ==> P 7)‘‘

> SIMP_CONV std_ss [] ‘‘?x. Q x /\ (x = 7) /\ P x‘‘
val it = |- (?x. Q x /\ (x = 7) /\ P x) <=> (Q 7 /\ P 7)‘‘

> SIMP_CONV std_ss [] ‘‘x > 7 ==> x > 5‘‘
Exception- UNCHANGED raised

> SIMP_CONV arith_ss [] ‘‘x > 7 ==> x > 5‘‘
val it = |- (x > 7 ==> x > 5) <=> T
```

Higher Order Rewriting

- the simplifier supports higher order rewriting
- this is often very handy
- for example it allows moving quantifiers around easily

Examples

```
> SIMP_CONV std_ss [FORALL_AND_THM] ``!x. P x /\ Q /\ R x``
val it = |- (!x. P x /\ Q /\ R x) <=>
            (!x. P x) /\ Q /\ (!x. R x)

> SIMP_CONV std_ss [GSYM RIGHT_EXISTS_AND_THM, GSYM LEFT_FORALL_IMP_THM]
   ``!y. (P y /\ (?x. y = SUC x)) ==> Q y``
val it = |- (!y. P y /\ (?x. y = SUC x) ==> Q y) <=>
            !x. P (SUC x) ==> Q (SUC x)
```

Context

- a great feature of the simplifier is that it can use context information
- by default simple context information is used like
  - the precondition of an implication
  - the condition of `if-then-else`
- one can configure which context to use via congruence rules
  - by using `CONJ_ss` one can easily use context of conjunctions
  - warning: using `CONJ_ss` can be slow
  - using other contexts is outside the scope of this lecture
- using context often simplifies proofs drastically
  - using `Rewrite` lib, often a goal needs to be split and a precondition moved to the assumptions
  - then `ASM_REWRITE_TAC` can be used
  - with `SIMP_TAC` there is no need to split the goal

Context Examples

```
> SIMP_CONV std_ss [] ‘‘((l = []) ==> P l) /\ Q l‘‘
val it = |- ((l = []) ==> P l) /\ Q l <=>
            ((l = []) ==> P []) /\ Q l

> SIMP_CONV arith_ss [] ‘‘if (c /\ x < 5) then (P c /\ x < 6) else Q c‘‘
val it = |- (if c /\ x < 5 then P c /\ x < 6 else Q c) <=>
            if c /\ x < 5 then P T else Q c:

> SIMP_CONV std_ss [] ‘‘P x /\ (Q x /\ P x ==> Z x)‘‘
Exception- UNCHANGED raised

> SIMP_CONV (std_ss++boolSimps.CONJ_ss) [] ‘‘P x /\ (Q x /\ P x ==> Z x)‘‘
val it = |- P x /\ (Q x /\ P x ==> Z x) <=> P x /\ (Q x ==> Z x)
```

Conditional Rewriting I

- perhaps the most powerful feature of the simplifier is that it supports conditional rewriting
- this means it allows **conditional** rewrite theorems of the form `|- cond ==> (t1 = t2)`
- if the simplifier finds a term `t1'` it can rewrite via `t1 = t2` to `t2'`, it tries to discharge the assumption `cond'`
- for this, it calls itself recursively on `cond'`
  - all the decision procedures and all context information is used
  - conditional rewriting can be used
  - to prevent divergence, there is a limit on recursion depth
- if `cond' = T` can be shown, `t1'` is rewritten to `t2'`
- otherwise `t1'` is not modified

Conditional Rewriting Example

- consider the conditional rewrite theorem
  `!l n. LENGTH l <= n ==> (DROP n l = [])`
- let's assume we want to prove
  `(DROP 7 [1;2;3;4]) ++ [5;6;7] = [5;6;7]`
- we can without conditional rewriting
  - show `|- LENGTH [1;2;3;4] <= 7`
  - use this to discharge the precondition of the rewrite theorem
  - use the resulting theorem to rewrite the goal
- with conditional rewriting, this is all automated

```
> SIMP_CONV list_ss [DROP_LENGTH_TOO_LONG]
    ``(DROP 7 [1;2;3;4]) ++ [5;6;7]``
val it = |- DROP 7 [1; 2; 3; 4] ++ [5; 6; 7] = [5; 6; 7]
```

- conditional rewriting often shortens proofs considerably

Conditional Rewriting Example II

Proof with Rewrite

```
prove (``(DROP 7 [1;2;3;4]) ++ [5;6;7] = [5;6;7]``,
`DROP 7 [1;2;3;4] = []` by (
  MATCH_MP_TAC DROP_LENGTH_TOO_LONG >>
  REWRITE_TAC[LENGTH] >>
  DECIDE_TAC
) >>
ASM_REWRITE_TAC[APPEND])
```

Proof with Simplifier

```
prove (``(DROP 7 [1;2;3;4]) ++ [5;6;7] = [5;6;7]``,
ASM_SIMP_TAC list_ss [])
```

Conditional Rewriting II

- conditional rewriting is a very powerful technique
- decision procedures and sophisticated rewrites can be used to discharge preconditions without cluttering proof state
- it provides a powerful search for theorems that apply
- however, if used naively, it can be slow
- moreover, to work well, rewrite theorems need to of a special form

Conditional Rewriting Pitfalls I

- if the pattern is too general, the simplifier becomes very slow
- consider the following, trivial but hopefully useful example

Looping example

```
> val my_thm = prove (``~P ==> (P = F)``, PROVE_TAC[])
> time (SIMP_CONV std_ss [my_thm]) ``P1 /\ P2 /\ P3 /\ ... /\ P10``
runtime: 0.84000s,    gctime: 0.02400s,     systime: 0.02400s.
Exception- UNCHANGED raised

> time (SIMP_CONV std_ss []) ``P1 /\ P2 /\ P3 /\ ... /\ P10``
runtime: 0.00000s,    gctime: 0.00000s,     systime: 0.00000s.
Exception- UNCHANGED raised
```

- notice that the rewrite is applied at plenty of places (quadratic in number of conjuncts)
- notice that each backchaining triggers many more backchainings
- each has to be aborted to prevent diverging
- as a result, the simplifier becomes very slow
- incidentally, the conditional rewrite is useless

Conditional Rewriting Pitfalls II

- good conditional rewrites `|- c ==> (l = r)` should mention only variables in `c` that appear in `l`
- if `c` contains extra variables `x1 ... xn`, the conditional rewrite engine has to search instantiations for them
- this mean that conditional rewriting is trying discharge the precondition `?x1 ... xn. c`
- the simplifier is usually not able to find such instances

Transitivity

```
> val P_def = Define ‘P x y = x < y‘;
> val my_thm = prove (‘‘!x y z. P x y ==> P y z ==> P x z‘‘, ...)
> SIMP_CONV arith_ss [my_thm] ‘‘P 2 3 /\ P 3 4 ==> P 2 4‘‘
Exception- UNCHANGED raised

(* However transitivity of < build in via decision procedure *)
> SIMP_CONV arith_ss [P_def] ‘‘P 2 3 /\ P 3 4 ==> P 2 4‘‘
val it = |- P 2 3 /\ P 3 4 ==> P 2 4 <=> T:
```

Conditional Rewriting Pitfalls III

- let's look in detail why SIMP_CONV did not make progress above

```
> set_trace "simplifier" 2;
> SIMP_CONV arith_ss [my_thm] ``P 2 3 /\ P 3 4 ==> P 2 4``
[468000]:   more context: |- !x y z. P x y ==> P y z ==> P x z
[468000]:   New rewrite: |- (?y. P x y /\ P y z) ==> (P x z <=> T)
...
[584000]:   more context:  [.] |- P 2 3 /\ P 3 4
[584000]:   New rewrite:  [.] |- P 2 3 <=> T
[584000]:   New rewrite:  [.] |- P 3 4 <=> T
[588000]:   rewriting P 2 4 with |- (?y. P x y /\ P y z) ==> (P x z <=> T)
[588000]:   trying to solve: ?y. P 2 y /\ P y 4
[588000]:   rewriting P 2 y with |- (?y. P x y /\ P y z) ==> (P x z <=> T)
[592000]:   trying to solve: ?y'. P 2 y' /\ P y' y
...
[596000]:   looping - cut
...
[608000]:   looping - stack limit reached
...
[640000]:   couldn't solve: ?y. P 2 y /\ P y 4
Exception- UNCHANGED raised
```

Conditional vs. Unconditional Rewrite Rules

- conditional rewrite rules are often much more powerful
- however, `Rewrite` lib does not support them
- for this reason there are often two versions of rewrite theorems

`drop` example

- `DROP_LENGTH_NIL` is a useful rewrite rule:
  `|- !l. DROP (LENGTH l) l = []`
- in proofs, one needs to be careful though to preserve exactly this form
  - one should not (partly) evaluate `LENGTH l` or modify `l` somehow
- with the conditional rewrite rule `DROP_LENGTH_TOO_LONG` one does not need to be as careful
  `|- !l n. LENGTH l <= n ==> (DROP n l = [])`
  - the simplifier can use simplify the precondition using information about `LENGTH` and even arithmetic decision procedures

Special Rewrite Forms

- some theorems given in the list of rewrites to the simplifier are used for special purposes
- there are marked functions that mark these theorems
  - `Once : thm -> thm` use given theorem at most once
  - `Ntimes : thm -> int -> thm` use given theorem at most the given number of times
  - `AC : thm -> thm -> thm` use given associativity and commutativity theorems for AC rewriting
  - `Cong : thm -> thm` use given theorem as a congruence rule
- these special forms are easy ways to add this information to a simpset
- it can be directly set in a simpset as well

Example Once

```
> SIMP_CONV pure_ss [Once ADD_COMM] ``a + b = c + d``
val it = |- (a + b = c + d) <=> (b + a = c + d)

> SIMP_CONV pure_ss [Ntimes ADD_COMM 2] ``a + b = c + d``
val it = |- (a + b = c + d) <=> (a + b = c + d)

> SIMP_CONV pure_ss [ADD_COMM] ``a + b = c + d``
Exception- UNCHANGED raised

> ONCE_REWRITE_CONV [ADD_COMM] ``a + b = c + d``
val it = |- (a + b = c + d) <=> (b + a = d + c)

> REWRITE_CONV [ADD_COMM] ``a + b = c + d``
... diverges ...
```

Stateful Simpset

- the simpset `srw_ss()` is maintained by the system
  - it is automatically extended by new type-definitions
  - theories can extend it via `export_rewrites`
  - libs can augment it via `augment_srw_ss`
- the stateful simpset contains many rewrites
- it is very powerful and easy to use

Example

```
> SIMP_CONV (srw_ss()) [] ``case [] of [] => (2 + 4)``
val it = |- (case [] of [] => 2 + 4 | v::v1 => ARB) = 6
```

Discussion on Stateful Simpset

- the stateful simpset is very powerful and easy to use
- however, results are hard to predict
- proofs using it unwisely are hard to maintain
- the stateful simpset can expand too much
  - bigger, harder to read proof states
  - high level arguments become hard to see
- whether to use the stateful simpset depends on personal proof style
- I advise at the beginning to not use `srw_ss`
- once you got a good intuition on how the simplifier works, make your own choice

Adding Own Conversions

- it is complicated to add arbitrary decision procedures to a simpset
- however, adding simple conversions is straightforward
- a conversion is described by a `stdconvdata` record

```
type stdconvdata = {
  name: string,     (* name for debugging *)
  pats: term list, (* list of patterns, when to try conv *)
  conv: conv        (* the conversion *)
}
```

- use `std_conv_ss` to create simpset-fragement

Example

```
val WORD_ADD_ss =
  simpLib.std_conv_ss
    {conv = CHANGED_CONV WORD_ADD_CANON_CONV,
     name = "WORD_ADD_CANON_CONV",
     pats = [‘‘words$word_add (w:’a word) y‘‘]}
```

Summary Simplifier

- the simplifier is HOL's main workhorse for automation
- it is very powerful
- conditional rewriting very powerful
  - here only simple examples were presented
  - experiment with it to get a feeling
- many advanced features not discussed here at all
  - using congruence rules
  - writing own decision procedures
  - rewriting with respect to arbitrary congruence relations

Warning

The simplifier is very powerful. Make sure you understand it and are in control when using it. Otherwise your proofs easily become lengthy, convoluted and hard to maintain.

Part XIV

Advanced Definition Principles

Relations

- a relation is a function from some arguments to `bool`
- the following example types are all types of relations:
  - `: 'a -> 'a -> bool`
  - `: 'a -> 'b -> bool`
  - `: 'a -> 'b -> 'c -> 'd -> bool`
  - `: ('a # 'b # 'c) -> bool`
  - `: bool`
  - `: 'a -> bool`
- relations are closely related to sets
  - `R a b c <=> (a, b, c) IN {(a, b, c) | R a b c}`
  - `(a, b, c) IN S <=> (\a b c. (a, b, c) IN S) a b c`

Relations II

- relations are often defined by a set of **rules**

Definition of Reflexive-Transitive Closure

The transitive reflexive closure of a relation `R : 'a -> 'a -> bool` can be defined as the least relation `RTC R` that satisfies the following rules:

$$\frac{\texttt{R x y}}{\texttt{RTC R x y}} \qquad \frac{}{\texttt{RTC R x x}} \qquad \frac{\texttt{RTC R x y} \qquad \texttt{RTC R y z}}{\texttt{RTC R x z}}$$

- if the rules are monoton, a least and a greatest fix point exists (Knaster-Tarski theorem)
- least fixpoints give rise to **inductive relations**
- greatest fixpoints give rise to **coinductive relations**

(Co)inductive Relations in HOL

- `(Co)IndDefLib` provides infrastructure for defining (co)inductive relations
- given a set of rules `Hol_(co)reln` defines (co)inductive relations
- 3 theorems are returned and stored in current theory
    - a rules theorem — it states that the defined constant satisfies the rules
    - a cases theorem — this is an equational form of the rules showing that the defined relation is indeed a fixpoint
    - a (co)induction theorem
- additionally a strong (co)induction theorem is stored in current theory

Example: Transitive Reflexive Closure

```
> val (RTC_REL_rules, RTC_REL_ind, RTC_REL_cases) = Hol_reln `
    (!x y.   R x y                            ==> RTC_REL R x y) /\
    (!x.                                          RTC_REL R x x) /\
    (!x y z. RTC_REL R x y /\ RTC_REL R x z ==> RTC_REL R x z)`

val RTC_REL_rules = |- !R.
  (!x y. R x y ==> RTC_REL R x y) /\ (!x. RTC_REL R x x) /\
  (!x y z. RTC_REL R x y /\ RTC_REL R x z ==> RTC_REL R x z)

val RTC_REL_cases = |- !R a0 a1.
  RTC_REL R a0 a1 <=>
  (R a0 a1 \/ (a1 = a0) \/ ?y. RTC_REL R a0 y /\ RTC_REL R a0 a1)
```

Example: Transitive Reflexive Closure II

```
val RTC_REL_ind = |- !R RTC_REL’.
  ((!x y. R x y ==> RTC_REL’ x y) /\ (!x. RTC_REL’ x x) /\
   (!x y z. RTC_REL’ x y /\ RTC_REL’ x z ==> RTC_REL’ x z)) ==>
  (!a0 a1. RTC_REL R a0 a1 ==> RTC_REL’ a0 a1)
```

```
> val RTC_REL_strongind = DB.fetch "-" "RTC_REL_strongind"

val RTC_REL_strongind = |- !R RTC_REL’.
  (!x y. R x y ==> RTC_REL’ x y) /\ (!x. RTC_REL’ x x) /\
  (!x y z.
      RTC_REL R x y /\ RTC_REL’ x y /\ RTC_REL R x z /\
      RTC_REL’ x z ==>
      RTC_REL’ x z) ==>
  ( !a0 a1. RTC_REL R a0 a1 ==> RTC_REL’ a0 a1)
```

Example: EVEN

```
> val (EVEN_REL_rules, EVEN_REL_ind, EVEN_REL_cases) = Hol_reln
  ‘(EVEN_REL 0) /\ (!n. EVEN_REL n ==> (EVEN_REL (n + 2)))‘;

val EVEN_REL_cases =
 |- !a0. EVEN_REL a0 <=> (a0 = 0) \/ ?n. (a0 = n + 2) /\ EVEN_REL n

val EVEN_REL_rules =
 |- EVEN_REL 0 /\ !n. EVEN_REL n ==> EVEN_REL (n + 2)

val EVEN_REL_ind = |- !EVEN_REL’.
   (!a0.
      EVEN_REL’ a0 ==>
      (a0 = 0) \/ ?n. (a0 = n + 2) /\ EVEN_REL’ n) ==>
   (!a0. EVEN_REL’ a0 ==> EVEN_REL a0)
```

- notice that in this example there is exactly one fixpoint
- therefore for these rule, the induction and coinductive relation coincide

Example: Dummy Relations

```
> val (DF_rules, DF_ind, DF_cases) = Hol_reln
  ‘(!n. DF (n+1) ==> (DF n))‘

> val (DT_rules, DT_coind, DT_cases) = Hol_coreln
  ‘(!n. DT (n+1) ==> (DT n))‘

val DT_coind =
  |- !DT’. (!a0. DT’ a0 ==> DT’ (a0 + 1)) ==> !a0. DT’ a0 ==> DT a0

val DF_ind =
  |- !DF’. (!n. DF’ (n + 1) ==> DF’ n) ==> !a0. DF a0 ==> DF’ a0

val DT_cases = |- !a0. DT a0 <=> DT (a0 + 1):
val DF_cases = |- !a0. DF a0 <=> DF (a0 + 1):
```

- notice that for both `DT` and `DF` we used essentially a non-terminating recursion
- `DT` is always true, i. e. `|- !n. DT n`
- `DF` is always false, i. e. `|- !n. ~(DF n)`

Quotient Types

- `quotientLib` allows to define types as quotients of existing types with respect to **partial equivalence relation**
- each equivalence class becomes a value of the new type
- partiality allows ignoring certain types
- `quotientLib` allows to lift definitions and lemmata as well
- details are technical and won't be presented here

Quotient Types Example

- let's assume we have an implementation of finite sets of numbers as binary trees with
  - type `binset`
  - binary tree invariant `WF_BINSET : binset -> bool`
  - constant `empty_binset`
  - add and member functions `add : num -> binset -> binset`, `mem : binset -> num -> bool`
- we can define a partial equivalence relation by

```
binset_equiv b1 b2 := (
  WF_BINSET b1 /\ WF_BINSET b2 /\
  (!n. mem b1 n <=> mem b2 n))
```

- this allows defining a quotient type of sets of numbers
- functions `empty_binset`, `add` and `mem` as well as lemmata about them can be lifted automatically

Quotient Types Summary

- quotient types are sometimes very useful
  - e. g. rational numbers are defined as a quotient type
- there is powerful infrastructure for them
- many tasks are automated
- however, the details are technical and won't be discussed here

Pattern Matching / Case Expressions

- pattern matching ubiquitous in functional programming
- pattern matching is a powerful technique
- it helps to write concise, readable definitions
- very handy and frequently used for interactive theorem proving in higher-order logic (HOL)
- however, it is not directly supported by HOL's logic
- representations in HOL
  - sets of equations as produced by `Define`
  - decision trees (printed as case-expressions)

TFL / Define

- we have already used top-level pattern matches with the TFL package
- Define is able to handle them
  - all the semantic complexity is taken care of
  - no special syntax or functions remain
  - no special rewrite rules, reasoning tools needed afterwards
- Define produces a set of equations
- this is the recommended way of using pattern matching in HOL

Example

```
> val ZIP_def = Define ‘(ZIP (x::xs) (y::ys) = (x,y)::(ZIP xs ys)) /\
                        (ZIP [] [] = [])‘
val ZIP_def = |- (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys) /\
                 (ZIP [] [] = [])
```

Case Expressions

- sometimes one does not want to use this compilation by TFL
  - one wants to use pattern-matches somewhere nested in a term
  - one might not want to introduce a new constant
  - one might want to avoid using TFL for technical reasons
- in such situations, case-expressions can be used
- their syntax is similar to the syntax used by SML

Example

```
> val ZIP_def = Define ‘ZIP xs ys = case (xs, ys) of
                                         (x::xs, y::ys) => (x,y)::(ZIP xs ys)
                                       | ([], []) => []‘
val ZIP_def = |- !ys xs. ZIP xs ys =
     case (xs,ys) of
       ([],[]) => []
     | ([],v4::v5) => ARB
     | (x::xs',[]) => ARB
     | (x::xs',y::ys') => (x,y)::ZIP xs' ys'
```

Case Expressions II

- the datatype package define case-constants for each datatype
- the parser contains a pattern compilation algorithm
- case-expressions are by the parser compiled to decision trees using case-constants
- pretty printer prints these decision trees as case-expressions again

Example

```
val ZIP_def = |- !ys xs. ZIP xs ys =
     pair_CASE (xs,ys)
       (\v v1.
          list_CASE v (list_CASE v1 [] (\v4 v5. ARB))
            (\x xs’. list_CASE v1 ARB (\y ys’. (x,y)::ZIP xs’ ys’))):
```

Case Expression Issues

- using case expressions feels very natural to functional programmers
- case-expressions allow concise, well-readable definitions
- however, there are also many drawbacks
- there is large, complicated code in the parser and pretty printer
  - this is outside the kernel
  - parsing a pretty-printed term can result in a non α-equivalent one
  - there are bugs in this code (see e. g. Issue #416 reported 8 May 2017)
- the results are hard to predict
  - heuristics involved in creating decision tree
  - results sometimes hard to predict
  - however, it is beneficial that proofs follow this internal, volatile structure

Case Expression Issues II

- technical issues
  - it is tricky to reason about decision trees
  - rewrite rules about case-constants needs to be fetched from `TypeBase`
    - alternative `srw_ss` often does more than wanted
  - partially evaluated decision-trees are not pretty printed nicely any more
- underspecified functions
  - decision trees are exhaustive
  - they list underspecified cases explicitly with value `ARB`
  - this can be lengthy
  - `Define` in contrast hides underspecified cases

Case Expression Example I

Partial Proof Script

```
val _ = prove (‘‘!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))‘‘,

ONCE_REWRITE_TAC [ZIP_def]
```

Current Goal

```
!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  (((case (l1,l2) of
       ([],[]) => []
     | ([],v4::v5) => ARB
     | (x::xs’,[]) => ARB
     | (x::xs’,y::ys’) => (x,y)::ZIP xs’ ys’) =
    []) <=> (l1 = []) /\ (l2 = []))
```

Case Expression Example IIa – partial evaluation

Partial Proof Script

```
val _ = prove (‘‘!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))‘‘,

ONCE_REWRITE_TAC [ZIP_def] >>
REWRITE_TAC[pairTheory.pair_case_def] >> BETA_TAC
```

Current Goal

```
!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  (((case l1 of
       [] => (case l2 of [] => [] | v4::v5 => ARB)
     | x::xs’ => case l2 of [] => ARB | y::ys’ => (x,y)::ZIP xs’ ys’) =
    []) <=> (l1 = []) /\ (l2 = []))
```

Case Expression Example IIb — following tree structure

Partial Proof Script

```
val _ = prove (‘‘!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))‘‘,

ONCE_REWRITE_TAC [ZIP_def] >>
Cases_on ‘l1‘ >| [
  REWRITE_TAC[listTheory.list_case_def]
```

Current Goal

```
!l2.
  (LENGTH [] = LENGTH l2) ==>
  (((case ([],l2) of
       ([],[]) => []
     | ([],v4::v5) => ARB
     | (x::xs’,[]) => ARB
     | (x::xs’,y::ys’) => (x,y)::ZIP xs’ ys’) =
    []) <=> (l2 = []))
```

Case Expression Summary

- case expressions are natural to functional programmers
- they allow concise, readable definitions
- however, fancy parser and pretty-printer needed
  - trustworthiness issues
  - sanity check lemmata advisable
- reasoning about case expressions can be tricky and lengthy
- proofs about case expression often hard to maintain
- therefore, use top-level pattern matching via `Define` if easily possible