Allowed beta decay
May 18, 2017

The study of nuclear beta decay provides information both about the nature of the weak interaction and about the structure of nuclear wave functions.

Outline

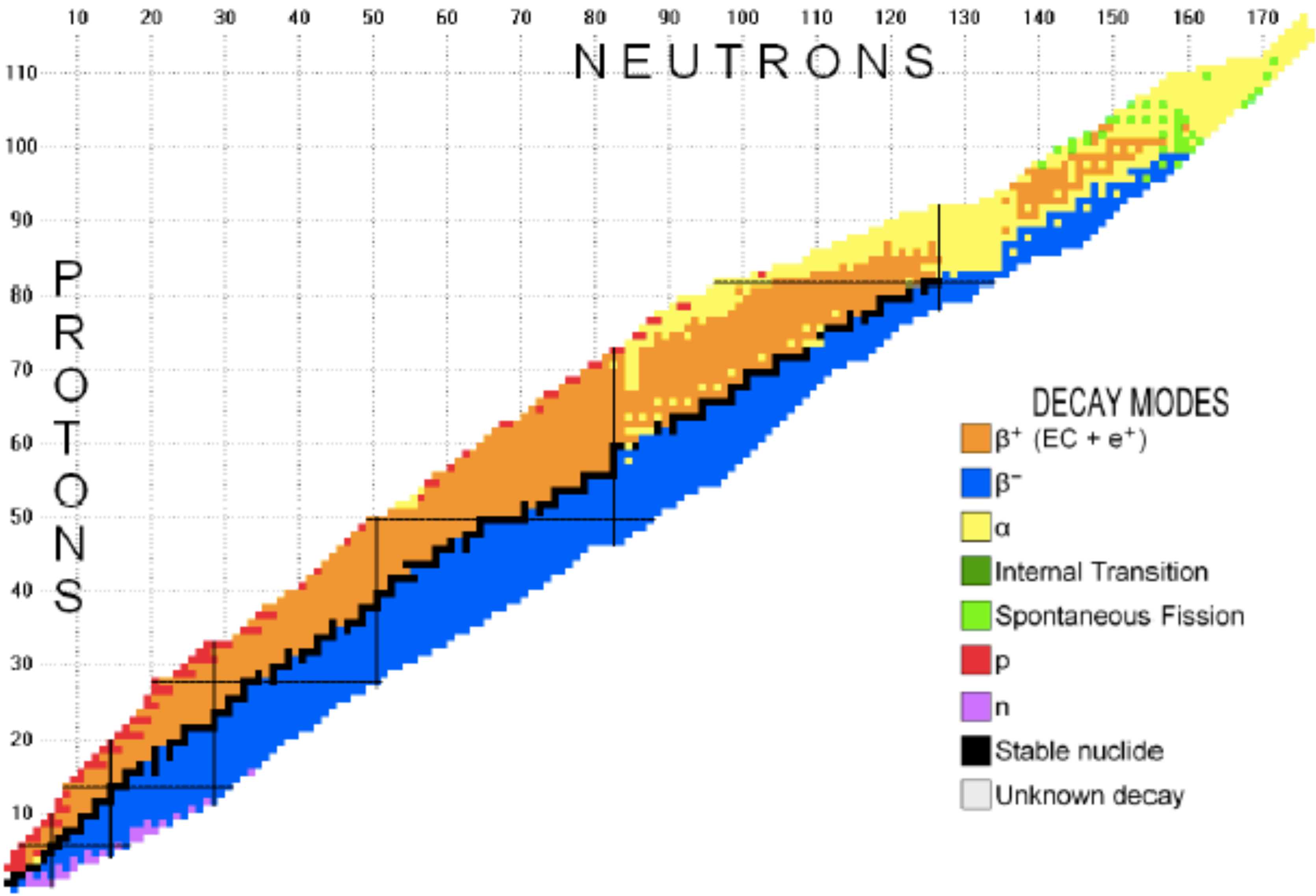
Basic concepts
beta decay, Q value, double beta decay

Operators and decay rates
Fermi and Gamow-Teller beta decays

Sum rules

^{14}C dating beta decay

Audi et al., Nucl. Phys. A 729 (2003) 3-128

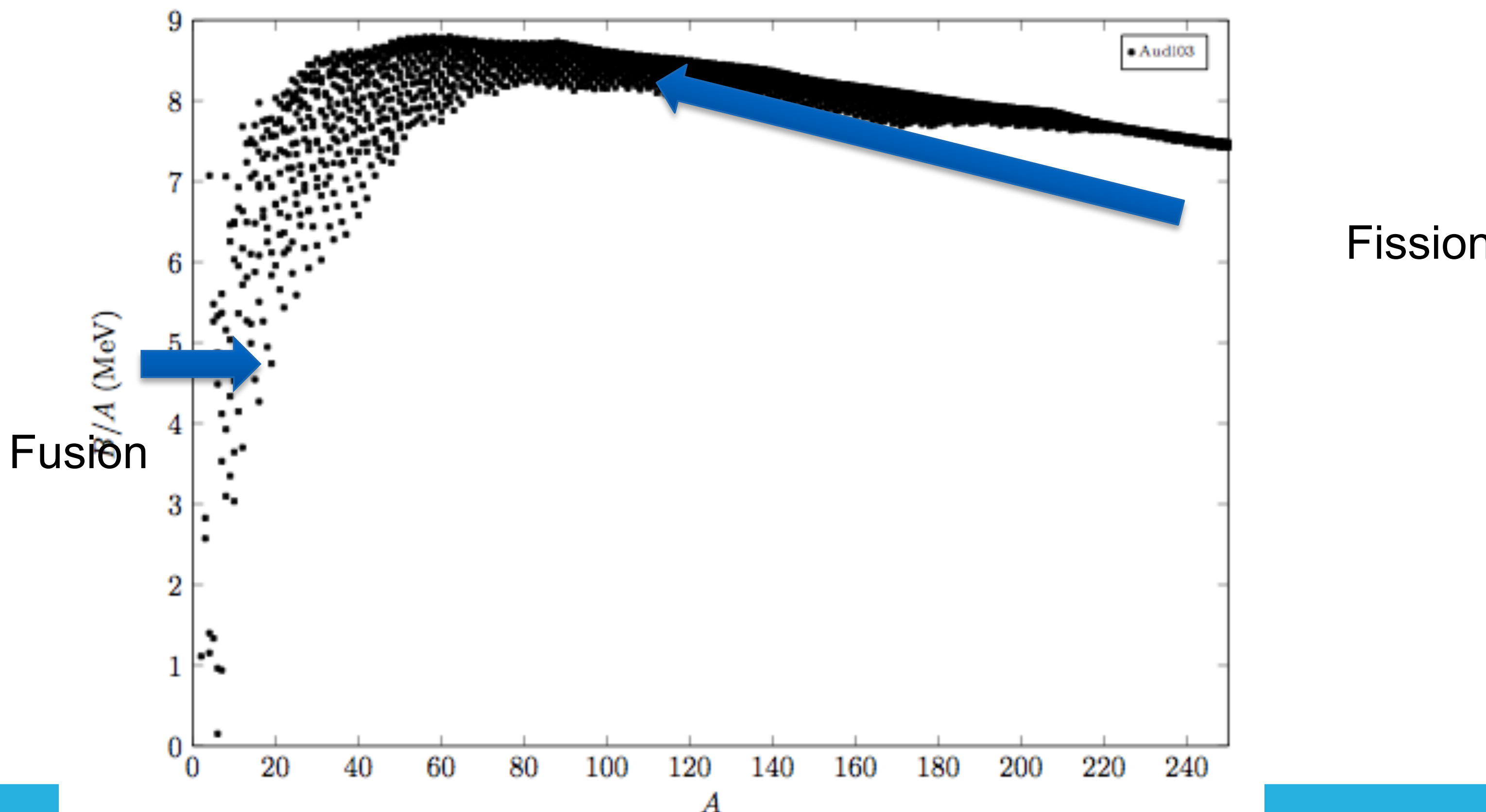


In the radioactive decay process

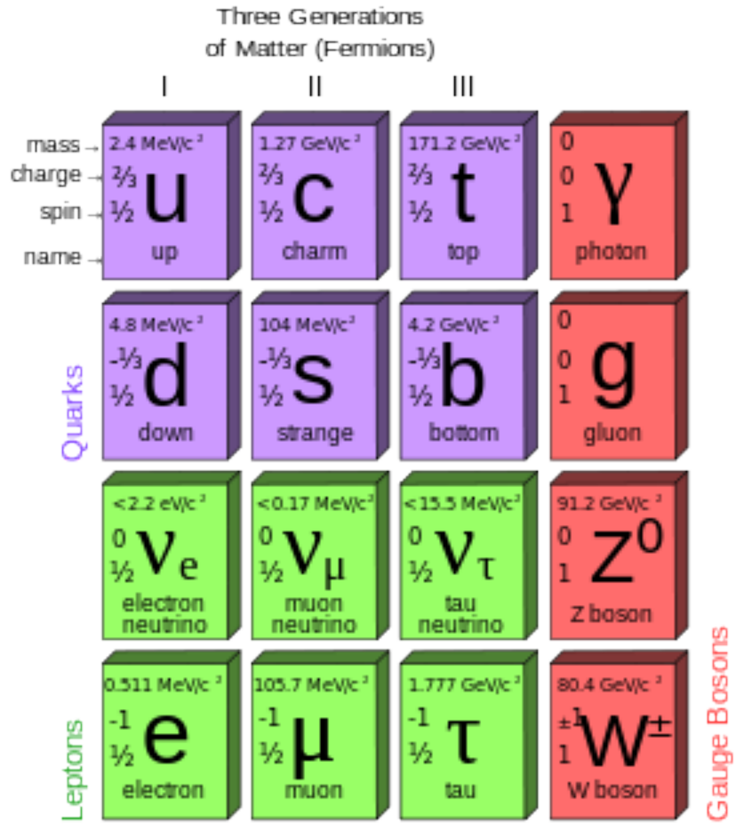
$$M \to D + C \tag{9.1}$$

where M is the mother nucleus, D the daughter and C the emitted cluster, energy conservation implies that the Q-value is

$$Q = B(D) + B(C) - B(M) \tag{9.2}$$



igure 9.1: Experimental binding energies per nucleon (in MeV).



The lighest nucleus that decays by beta emission is the free neutron, which undergoes the spontaneous process

$$n \to p + e^- + \bar{\nu} \qquad (11.1)$$

with a half life of about 10.3 minutes. In this equation n (p) is a neutron (proton), e^- is an electron and $\bar{\nu}$ is an antineutrino. Antiparticles have the same mass and spin as the corresponding particles, but opposite charge and spin projection. Thus the antiparticle of the electron (e^-) is the positron (e^+). The neutron is heavier than the proton, which explains why it is unstable.

Q-values in β^-, β^+ and electron capture decays

The decay energy associated to the process (11.1), which is called "the Q-value", is given by

$$Q_{\beta^-} = m_n c^2 - m_p c^2 - m_e c^2 \tag{11.2}$$

Since $m_n c^2$=939.566 MeV, $m_p c^2$=938.272 MeV and $m_e c^2$=0.511 MeV, the Q-value is Q_{β^-}=0.783 MeV. This is the kinetic energy shared by the three particles in the exit channel of Eq. (11.1).

For the β^+ decay, i. e.

$$p \rightarrow n + e^+ + \nu \tag{11.3}$$

it is a positron and a neutrino that are emitted. Since the masses of the antiparticles are the same as the ones of the corresponding particles one has, $Q_{\beta^+} = m_p c^2 - m_n c^2 - m_e c^2$=-1.805 MeV, which implies that this decay, with negative kinetic energies for the outgoing particles, is forbidden. Or, in other words, for the process (11.3) to take place it is necessary that energy is provided from the outside.

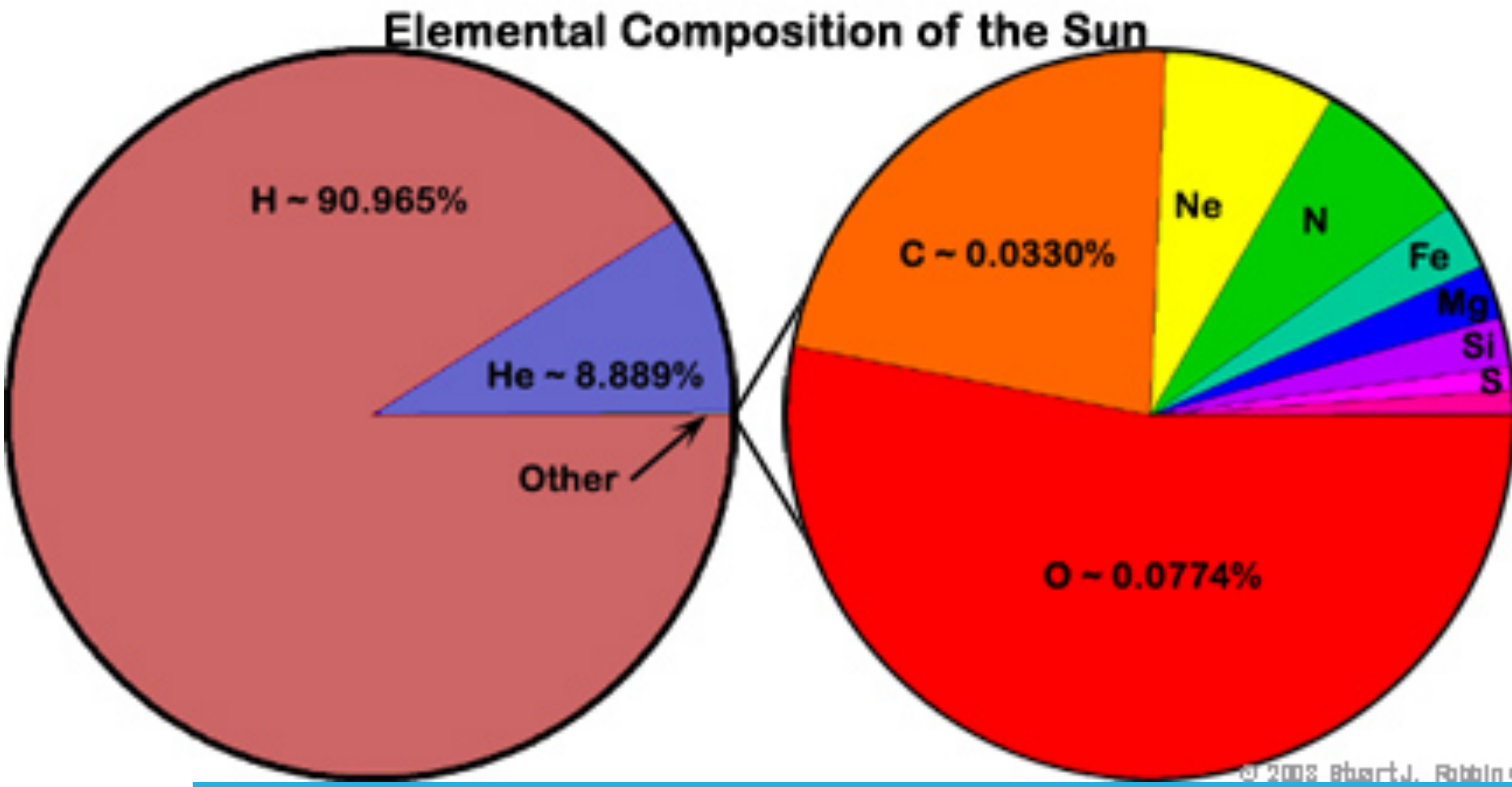
Another form of decay is electron capture (EC) by a proton in the process,

$$p + e^- \rightarrow n + \nu \tag{11.4}$$

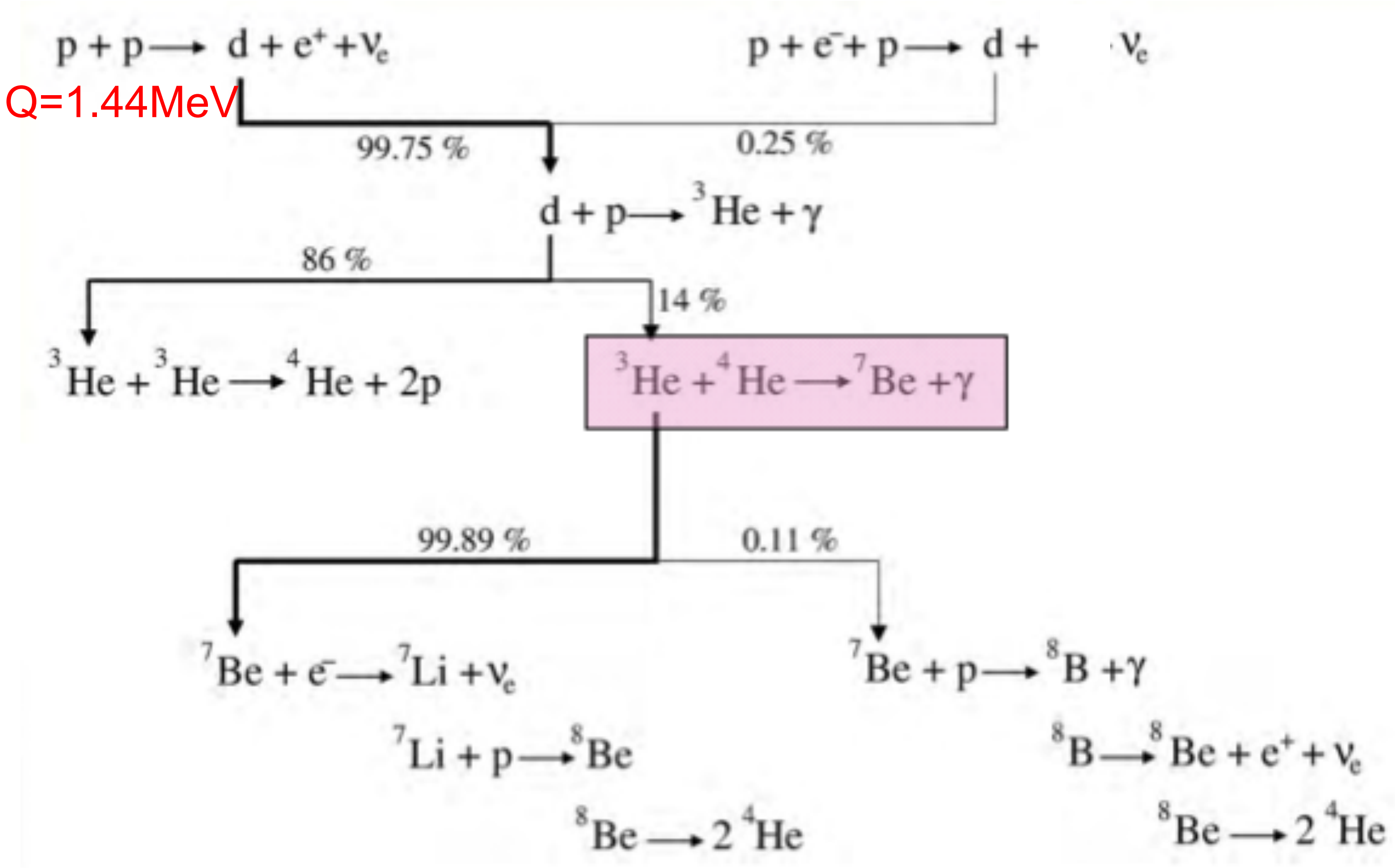
for which the Q-value is $Q_{EC} = m_p c^2 + m_e c^2 - m_n c^2 -$=-0.783 MeV, which is also a forbidden transition.

Stars are formed through the presence of free protons in space which clump together under the influence of the gravitational field.

In the center of the stars thus formed the protons are concentrated in a high temperature and high density environment. In this environment the protons interact with each other to produce heavier isotopes in a process that goes on until the protons are depleted.



The pp chain



At the end of the reactions only alpha particles left.

Nuclei are composed of protons and neutrons bound together by the strong interaction. In the beta decay of nuclei, a given initial nuclear state ${}^{A_i}Z_i$ is converted into the ground state or an excited state of the final nucleus ${}^{A_f}Z_f$, where $Z_f = Z_i \pm 1$. The transition rate for nuclear beta decay is determined by the Q_β value or energy release and the structure of the initial and final nuclear states.

$$\beta^-, \qquad {}^{A}Z \to {}^{A}(Z+1) + e^- + \bar{\nu}_e.$$

The Q value for β^- decay is given in terms of nuclear masses M and nuclear binding energies BV by

$$Q(\beta^-) = [M(A,Z) - M(A,Z+1) - m_e]c^2 = [M(A,Z) - M(A,Z+1)]c^2$$

$$= B(A,Z+1) - B(N,Z) + \delta_{nH},$$

$$\delta_{nH} = \Delta_n c^2 - \Delta_H c^2 = 0.782 \text{ MeV}$$

$$m_e c^2 = 0.511 \text{ MeV}$$

For the nuclear β^+ decay it is,

$$(Z, N) \rightarrow (Z-1, N+1) + e^+ + \nu \tag{11.7}$$

and $Q_{\beta^+} = M(Z, N) - M(Z-1, N+1) - m_e c^2$, which can also be positive or negative according to the structure of the nuclei.

Finally, the nuclear electron capture occurs when one of the electrons orbiting around the nucleus is absorbed by a proton in the nucleus. The proton is transformed into a neutron and an electrino is emitted following the process (11.4), i. e.

$$(Z, N) + e^- \rightarrow (Z-1, N+1) + \nu \tag{11.8}$$

with $Q_{EC} = M(Z, N) + m_e c^2 - M(Z-1, N+1)$

Double beta decay

Nuclear double-beta decay takes place in situations where a nucleus is energetically stable to single-beta decay but unstable to the simultaneous emission of two electrons (or two positrons).

There are two types of double-beta decay: the standard $(2e, 2\nu)$ type in which two neutrinos are emitted:

$$^{A}Z \rightarrow {}^{A}(Z+2) + 2e^{-} + 2\bar{\nu}_e,$$

or

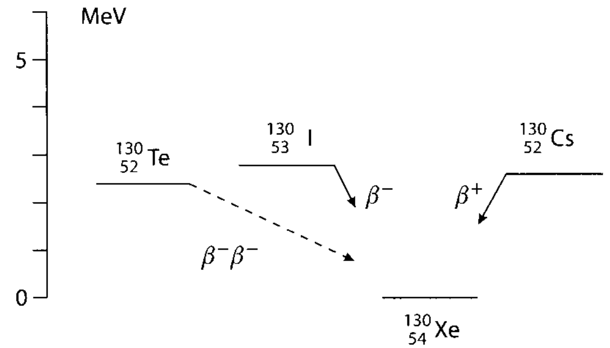
$$^{A}Z \rightarrow {}^{A}(Z-2) + 2e^{+} + 2\nu_e,$$

and the $(2e, 0\nu)$ type in which no neutrinos are emitted:

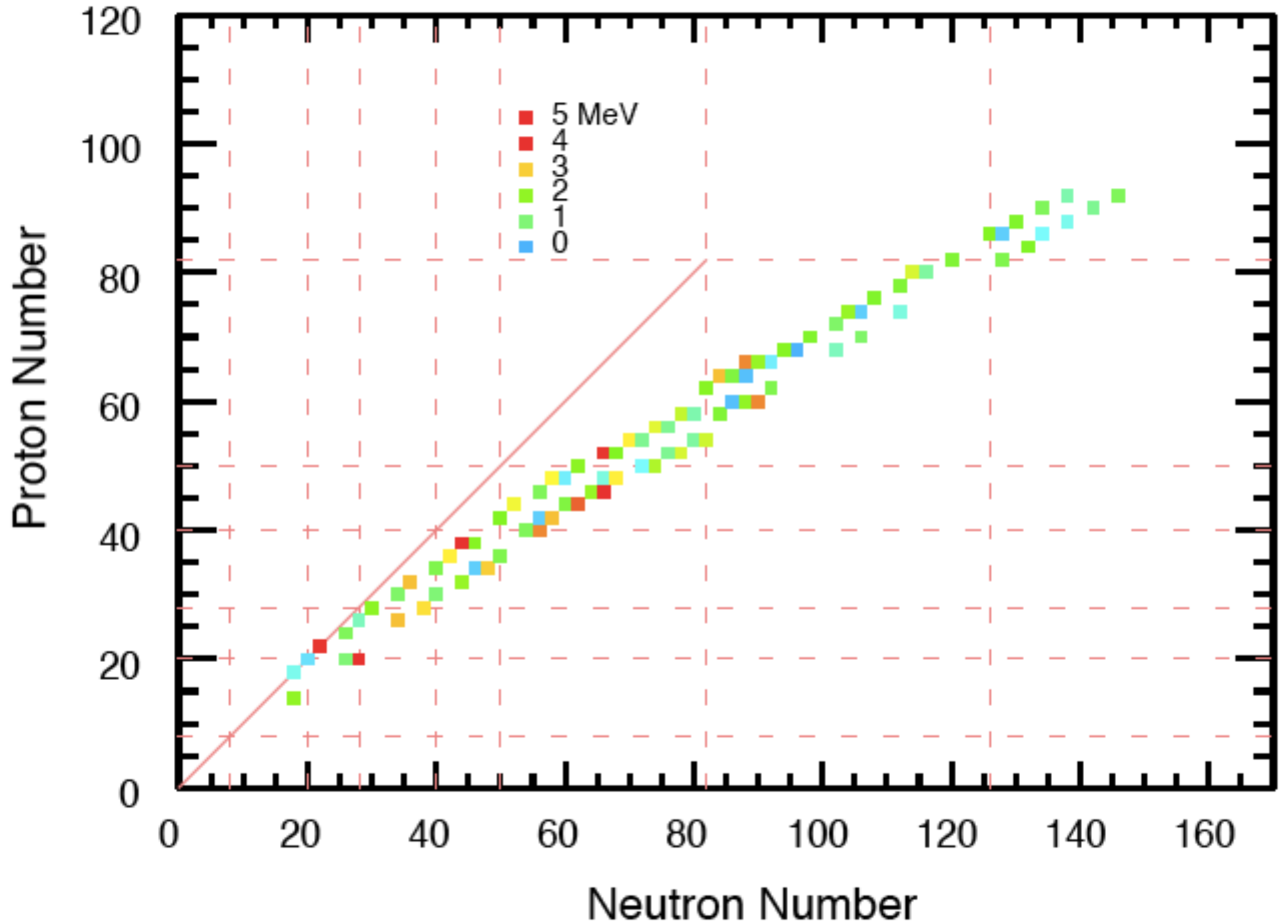
$$^{A}Z \rightarrow {}^{A}(Z+2) + 2e^{-},$$

or

$$^{A}Z \rightarrow {}^{A}(Z-2) + 2e^{+}.$$



double beta decay Q values



Reminder: Gamma transition

ansition is given as $1/T_{i,f,\lambda}$ and the partia

$$t_{1/2}^{if} = \frac{\ln 2}{T_{i,f,\lambda}}.$$

ven (initial) state is given as

$$\frac{1}{t_{1/2}} = \sum_f \frac{1}{t_{1/2}^{if}}.$$

One-body transition operator in the coupled form

As mentioned before, a one-body operator can be written as

$$\mathcal{O}^{\lambda}_{\mu} = \sum_{\alpha\beta} \langle\alpha|\mathcal{O}^{\lambda}_{\mu}|\beta\rangle a^{\dagger}_{\alpha} a_{\beta}, \tag{7.24}$$

where the Greek letters denote the single-particle states $|nljm\rangle$. We use j to denote the state $|nlj\rangle$. One has,

$$\begin{aligned}\mathcal{O}^{\lambda}_{\mu} =& \sum_{j_\alpha j_\beta} \sum_{m_\alpha m_\beta} \langle\alpha|\mathcal{O}^{\lambda}_{\mu}|\beta\rangle a^{\dagger}_{\alpha} a_{\beta} \\ =& \sum_{j_\alpha j_\beta} \langle j_\alpha||\mathcal{O}^{\lambda}_{\mu}||j_\beta\rangle \sum_{m_\alpha m_\beta} (-1)^{j_\alpha - m_\alpha} \begin{pmatrix} j_\alpha & \lambda & j_\beta \\ -m_\alpha & \mu & m_\beta \end{pmatrix} a^{\dagger}_{\alpha} a_{\beta} \\ =& \sum_{j_\alpha j_\beta} \langle j_\alpha||\mathcal{O}^{\lambda}_{\mu}||j_\beta\rangle \frac{[a^{\dagger}_{\alpha} \otimes \tilde{a}_{\beta}]^{\lambda}_{\mu}}{\sqrt{2\lambda+1}},\end{aligned} \tag{7.25}$$

where $\tilde{a}_{jm} = (-1)^{j+m} a_{j,-m}$.

where $\tilde{a}_{jm} = (-1)^{j+m} a_{j,-m}$. For the one-body transition density we have

$$\langle j_f m_f | [a^{\dagger}_{\alpha} \otimes \tilde{a}_{\beta}]^{\lambda}_{\mu} | j_i m_i \rangle = \delta_{\alpha f} \delta_{\beta i} (-1)^{j_i - m_i} < j_f m_f j_i - m_i | \lambda\mu > .$$

The reduced single-particle matrix element for $E\lambda$ operator is given by:

$$\langle j_a||\mathcal{O}(E\lambda)||j_b\rangle = (-1)^{j_a+1/2}\frac{1+(-1)^{l_a+\lambda+l_b}}{2}\sqrt{\frac{(2j_a+1)(2\lambda+1)(2j_b+1)}{4\pi}} \times \begin{pmatrix} j_a & \lambda & j_b \\ 1/2 & 0 & -1/2 \end{pmatrix} \langle j_a|r^\lambda|j_b\rangle e_{t_z} e \tag{7.42}$$

For the M1 operator the radial matrix element is:

$$\langle j_a|r^0|j_b\rangle = \delta_{n_a,n_b} \tag{7.48}$$

and the reduced single-particle matrix element simplify to:

$$\begin{aligned}\langle j_a||\mathcal{O}(M1,s)||j_b\rangle &= \sqrt{\frac{3}{4\pi}}\langle j_a||\boldsymbol{s}||j_b\rangle\delta_{n_a,n_b}g^s_{t_z}\mu_N \\ &= \sqrt{\frac{3}{4\pi}}(-1)^{l_a+j_a+3/2}\sqrt{(2j_a+1)(2j_b+1)} \\ &\times \begin{Bmatrix} 1/2 & 1/2 & 1 \\ j_b & j_a & l_a \end{Bmatrix}\langle s||\boldsymbol{s}||s\rangle\delta_{l_a,l_b}\delta_{n_a,n_b}g^s_{t_z}\mu_N\end{aligned} \tag{7.49}$$

Introducing the analogues of the Pauli matrices into isospin space

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

one defines the isospin vector operator as

$$\boldsymbol{t} = \tfrac{1}{2}\boldsymbol{\tau}.$$

it follows that the components t_x, t_y and t_z obey the commutation relations of an angular momentum

$$[t_x, t_y] = i\, t_z$$

$$[\boldsymbol{t}^2, t_z] = 0$$

and that the eigenvalues of $\boldsymbol{t}^2$ are given by

$$\boldsymbol{t}^2 \Rightarrow t(t+1).$$

It should be stressed that *the isospin bears no relation to ordinary space.*

In analogy with angular momentum one can define raising and lowering operators as

$$t_+ = \frac{1}{2}\tau_+ = \frac{1}{2}(\tau_x + i\tau_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$t_- = \frac{1}{2}\tau_- = \frac{1}{2}(\tau_x - i\tau_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The operators $\tau_\pm$ that transform a proton state into a neutron state and *vice versa*

Direct application of the explicit matrix representations to the isospinors yields the relations

$$t_+\phi_p = \phi_n\,, \qquad t_+\phi_n = 0\,, \qquad t_-\phi_p = 0\,, \qquad t_-\phi_n = \phi_p$$

The weak interaction

We assume a system of non-relativistic spinless nucleons (zeroth-order approximation) and a point interaction represented by **the** Hamiltonian

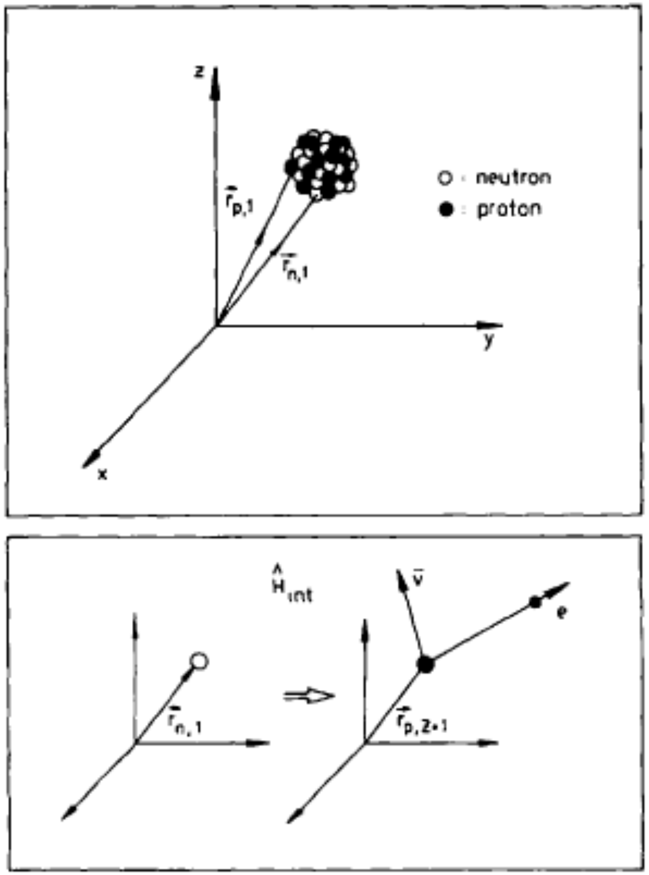
$$H_{\rm int} = g\delta(\vec{r}_{\rm n} - \vec{r}_{\rm p})\delta(\vec{r}_{\rm n} - \vec{r}_{\rm e^-})\delta(\vec{r}_{\rm n} - \vec{r}_{\bar{\nu}_{\rm e}})\left(\hat{O}({\rm n} \rightarrow {\rm p})\right).$$

The parent and daughter wave functions

$$\psi_{\rm P}(\vec{r}_{p.1},\ \vec{r}_{p.2}, \ldots, \vec{r}_{p.Z};\ \vec{r}_{n.1}, \ldots, \vec{r}_{n.N})$$

$$\psi_{\rm D}(\vec{r}_{p.1},\ \vec{r}_{p.2}, \ldots, \vec{r}_{p.Z}, \vec{r}_{p.Z+1};\ \vec{r}_{n.2}, \ldots, \vec{r}_{n.N}).$$

Schematic illustration of the neutron-to-proton transition,



The reduced matrix elements

$$M'_{fi} = \int \psi^*_{\rm D}(\vec{r}_{p.1}, \ldots \vec{r}_{p.Z+1};\ \vec{r}_{n.2}, \ldots \vec{r}_{n.N})\psi^*_{\rm e^-}(\vec{r}_{\rm e^-})\psi^*_{\bar{\nu}_{\rm e}}(\vec{r}_{\bar{\nu}_{\rm e}}).\ H_{\rm int}$$
$$\psi_{\rm P}(\vec{r}_{p.1}, \ldots \vec{r}_{p.Z};\ \vec{r}_{n.1}, \ldots \vec{r}_{n.N})\,{\rm d}\vec{r}_{\rm e^-}\,{\rm d}\vec{r}_{\bar{\nu}_{\rm e}}\,{\rm d}\vec{r}_i\,.$$

state $|\beta(P)\rangle$. The nuclear matrix element should thus have the form

$$\mathcal{M} = \int \Psi^*_{\beta(P)}(\mathbf{r})\mathcal{O}\Psi_{\alpha(N)}(\mathbf{r})d\mathbf{r} \tag{11.9}$$

where $\mathcal{O}$ is some operator that one has to introduce to explain the experimental data.

The transition probability per unit of time is given by the "golden rule" of quantum mechnics, i. e. it is the square of the transition matrix element times the density of states, which in our case is,

$$T = \frac{2\pi}{\hbar}|N_e N_\nu \mathcal{M}|^2 \frac{d\mathcal{N}}{dE} \tag{11.10}$$

We will assume that the system is contained in a box of volume V and, therefore, $N_e = N_\nu = 1/\sqrt{V}$. $d\mathcal{N}/dE$ is the density of final states. A change in E implies a change in the energy of the electron, which determines any change in the energy of the neutrino. This is because the transition energy $E_t = E_e + E_\nu$ is a constant, i. e. $dE_t = 0$, and one has $dE = dE_e = -dE_\nu$. One can then write the density of states as $d\mathcal{N}/dE = d\mathcal{N}/dE_e$.

$$\psi_{\mathrm{P}}(\vec{r}_i) = \prod_{i=1}^{Z} \varphi_{\mathrm{P}}(\vec{r}_{p.i}) . \prod_{j=1}^{N} \varphi_{\mathrm{P}}(\vec{r}_{n.j})$$

$$\psi_{\mathrm{D}}(\vec{r}_i) = \prod_{i=1}^{Z+1} \varphi_{\mathrm{D}}(\vec{r}_{p.i}) . \prod_{j=2}^{N} \varphi_{\mathrm{D}}(\vec{r}_{n.j}),$$

$$M'_{fi} = g \int \varphi^*_{\mathrm{D}}(\vec{r})_{\mathrm{p}} \psi^*_{\mathrm{e}^-}(\vec{r}) \psi^*_{\bar{\nu}_e}(\vec{r}) \hat{O}(n \to p) \varphi_{\mathrm{P}}(\vec{r})_{\mathrm{n}} \mathrm{d}\vec{r}.$$

Considering plane waves to describe the outgoing electron and antineutrino

$$M'_{fi} = g \int \varphi^*_{\mathrm{D}}(\vec{r})_{\mathrm{p}} e^{\mathrm{i}(\vec{k}_{e^-} + \vec{k}_{\bar{\nu}_e}).\vec{r}} \hat{O}(n \to p) \varphi_{\mathrm{P}}(\vec{r})_{\mathrm{n}} \mathrm{d}\vec{r}.$$

An expansion technique similar to EM transitions

$$M'_{fi} = g\left[\int \varphi^*_D(\vec{r})_p \hat{O}(n \to p)\varphi_P(\vec{r})_n d\vec{r} + i\left(\vec{k}_{e^-} + \vec{k}_{\bar{\nu}_e}\right) \cdot \int \varphi^*_D(\vec{r})_p \vec{r}\,\hat{O}(n \to p)\varphi_P(\vec{r})_n d\vec{r} + \ldots\right].$$

The **first term means that a neutron is transformed into a proton and the integral just**
measures the overlap between both nuclear, single-particle wave functions. This term is only present if the parities of both the parent and daughter single-particle wave functions are identical. For the second term, because of the ***r factor, the wave functions must have*** opposite parity, etc .

Angular momentum selection can also be deduced ***using*** the expansion of a plane wave into spherical harmonics

$$e^{i(\vec{k}_{e^-}+\vec{k}_{\bar{\nu}_e})\cdot\vec{r}} = \sum_{L,M}(4\pi)i^L j_L(kr) Y_L^M\left(\widehat{k_e + k_{\bar{\nu}_e}}\right).Y_L^{M^*}(\hat{r}),$$

where $k \equiv \left|\vec{k}_{e^-} + \vec{k}_{\bar{\nu}_e}\right|$ and $\hat{r} \equiv (\theta_r, \varphi_r)$, denote the angular variables.

For the spherical Bessel function, we have

$$j_n(kr) = \frac{-i^{n}}{2}\int_0^{\pi} \sin(\theta)\, d\theta\, e^{ikr\cos(\theta)}\, P_n(\cos(\theta))$$

$$\int \varphi_D^*(\vec{r})_p \hat{O}(n \to p) Y_L^{M^*}(\hat{r}) \varphi_P(\vec{r})_n j_L(kr) d\vec{r}.$$

The types of beta decay can be classified by the angular momenta carried away by the electron and neutrino. The most important are those for dℓ=0 which are referred to as “allowed” beta decay.

$$\varphi_D(\vec{r}) = R_D(r) Y_{L_D}^{M_D}(\hat{r}) \qquad \varphi_P(\vec{r}) = R_P(r) Y_{L_P}^{M_P}(\hat{r}).$$

This reduces the particular L-matrix element into a product of a pure radial and an angular
part

$$\int Y_{L_D}^{M_D^*}(\hat{r})\, Y_L^{M^*}(\hat{r}) Y_{L_P}^{M_P}(\hat{r})\, d\Omega . \int R_D(r) j_L(kr) R_P(r) r^2\, dr .$$

Introducing intrinsic spin

$$\bar{M}'_{fi} = \int \sum \psi_D^* \hat{\sigma}\, \psi_P . \psi_{e^-}^* \hat{\sigma}\, \psi_{\nu_e}\, d\vec{r},$$

$$\vec{J}_P = \vec{J}_D + \vec{J}_\beta \qquad (\vec{J}_\beta = \vec{L}_\beta + \vec{S}_\beta) \qquad \pi_P = \pi_D(-1)^{L_\beta},$$

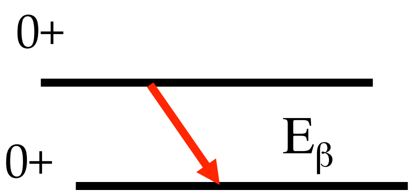
$$\vec{S}_\beta = \vec{0}, \vec{1}$$

There are two type of allowed beta decay − Fermi (F) and Gamow- Teller (GT). The operator associated with Fermi decay is proportional to the isospin raising and lowering operator. The operator associated with Gamow-Teller decay also contains the nucleon spin operator. Gamow-Teller beta decay goes in general to many final states since the total spin S is not a good quantum number.

Classification of allowed decay

$$(\pi_i \pi_f = +1)$$

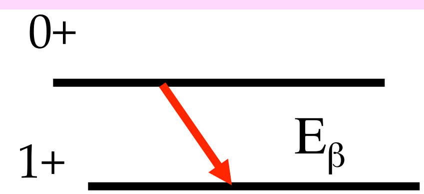
Fermi



$$\Delta I = \left| I_i - I_f \right| \equiv 0$$

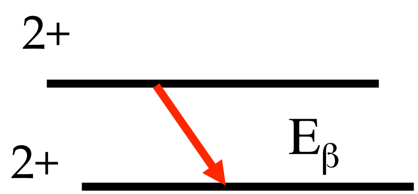
$$L_\beta = 0 \qquad S_\beta = 0 \downarrow\uparrow$$

Gamow-Teller



$$\Delta I = \left| I_i - I_f \right| \equiv 1$$

$$L_\beta = 0 \quad S_\beta = 1 \uparrow\uparrow or \downarrow\downarrow$$



mixed Fermi & Gamow-Teller

$$\Delta I = \left| I_i - I_f \right| \equiv 0 \qquad I_i \neq 0$$

I. Allowed transitions ($\vec{L}_\beta = 0,\ \pi_P = \pi_D$)

Fermi-type ($\vec{S}_\beta = \vec{0}$)

$\vec{J}_P = \vec{J}_D$

$|\Delta J| = 0$

$0^+ \to 0^+$: superallowed

Gamow–Teller type ($\vec{S}_\beta = \vec{1}$)

$\vec{J}_P = \vec{J}_D + \vec{1}$

$|\Delta J| = 0, 1$: no $0^+ \to 0^+$

$0^+ \to 1^+$: unique Gamow–Teller

II. 1st forbidden transitions ($\vec{L}_\beta = \vec{1},\ \pi_P = -\pi_D$)

Fermi-type ($\vec{S}_\beta = \vec{0}$)

$\vec{J}_P = \vec{J}_D + \vec{1}$

Gamow–Teller type ($\vec{S}_\beta = \vec{1}$)

$$\vec{J}_P = \vec{J}_D + \underbrace{\vec{1} + \vec{1}}_{\vec{0},\vec{1},\vec{2}}$$

Historically one combines the partial half-life for a particular decay with the calculated phase-space factor f to obtain an "ft" value given by

$$ft_{1/2} = \frac{C}{[B(F_{\pm}) + (g_A/g_V)^2 B(GT_{\pm})]}$$

where

$$C = \frac{\ln(2)\, K_o}{(g_V)^2} = 6170$$

The transition probability (8.10) for the electron in a range of momentum dp_e becomes,

$$T dp_e = \frac{|\mathcal{M}|^2}{2\pi^3\hbar^7 c^3}(E_e^{max} - E_e)^2 p_e^2 dp_e \tag{8.14}$$

notice that the volumen V has been cancelled out in this expression, which implies that its exact value is irrelevant, as it should be. Calling $P_e = p_e^{max}$ one has $E_e^{max} = \sqrt{m_e^2c^4 + c^2P_e^2}$. Replacing this in Eq. (8.14) one obtains,

$$T dp_e = \frac{|\mathcal{M}|^2}{2\pi^3\hbar^7 c^3}\left(\sqrt{m_e^2c^4 + c^2P_e^2} - \sqrt{m_e^2c^4 + c^2p_e^2}\ \right)^2 p_e^2 dp_e \tag{8.15}$$

$$f(P_e)m_e^5c^7 = \int_0^{P_e} F(Z_f, p_e)\left(\sqrt{m_e^2c^4 + c^2P_e^2} - \sqrt{m_e^2c^4 + c^2p_e^2}\ \right)^2 p_e^2 dp_e$$

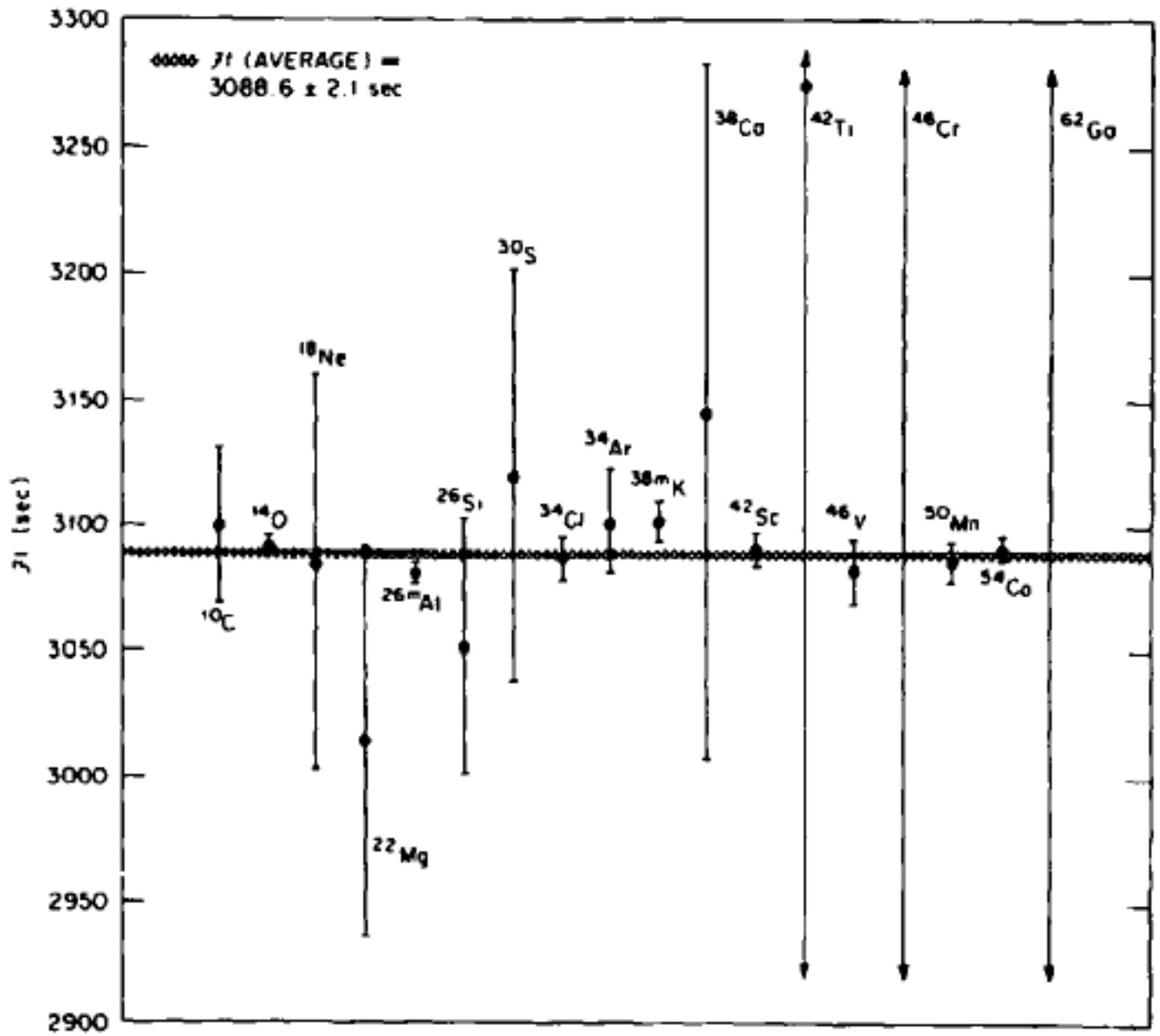
Table 8.1: Values of $log_{10}ft$ and the corresponding restrictions upon the angular momentum transfer ΔJ and parity change $\Delta\pi$.

$log_{10}ft$	ΔJ	l_β	$\Delta\pi$	Decay type
2.9 - 3.7	0	0	no	Supperallowed (Fermi)
3.8 - 6	0,1	0	no	Allowed Gamow-Teller & Fermi
6 - 10	0,1,2	0,1	yes	First forbidden
10 - 13	1,2,3	1,2	no	Second forbidden
17-19	2,3,4	2,3	yes	Third forbidden

Table 3-2 Classification of Beta Transitions and Selection Rules

Type	l	ΔI	$\Delta\Pi$	Log ft^a	Log $[(W_0^2-1)^{\Delta I-1} ft]$	Examples
Superallowed	0	0 or 1	No	3		^3H, ^{23}Mg
Allowed (normal)	0	0 or 1	No	4–7		^{35}S, ^{69}Zn
Allowed (l-forbidden)	0	1	No	6–12		^{14}C, ^{32}P
First forbidden	1	0 or 1	Yes	6–15		^{111}Ag, ^{143}Ce
First forbidden (unique)	1	2	Yes	9–13	~10	^{38}Cl, ^{90}Sr
Second forbidden	2	2	No	11–15		^{36}Cl, ^{135}Cs
Second forbidden (unique)	2	3	No	13–18	~15	^{10}Be, ^{22}Na
Third forbidden	3	3	Yes	17–19		^{87}Rb
Third forbidden (unique)	3	4	Yes		~21	^{40}K
Fourth forbidden	4	4	No	~23		^{115}In
Fourth forbidden (unique)	4	5	No		~28	

Superallowed fT values for the various decays.



The operators for Fermi and Gamow-Teller beta decays in terms of sums over nucleons are:

$$\mathcal{O}(F_\pm) = \sum_k t_{k\pm},$$

$$\mathcal{O}(GT_\pm) = \sum_k \sigma_k t_{k\pm}.$$

$$B_{i,f}(F_\pm) = \frac{|< f||\mathcal{O}(F_\pm)||i >|^2}{(2J_i + 1)},$$

$$B_{i,f}(GT_\pm) = \frac{|< f||\mathcal{O}(GT_\pm)||i >|^2}{(2J_i + 1)}$$

The matrix elements are reduced in orbital space and the (2i+1) factor comes from the average over initial M states.

case it is $\Psi_{\alpha(N)}(\mathbf{r}) = R_{n_\alpha l_\alpha j_\alpha}(r)\left[Y_{l_\alpha}(\hat{r})\chi_{1/2}\right]_{j_\alpha m_\alpha}$ and the same for $\Psi^*_{\beta(P)}(\mathbf{r})$. Integrating over the angles the matrix element becomes

$$\mathcal{M} = g_V \delta_{l_\alpha l_\beta} \delta_{j_\alpha j_\beta} \int R^*_{n_\beta l_\beta j_\beta(P)}(r) R_{n_\alpha l_\alpha j_\alpha(N)}(r) r^2 dr \tag{8.22}$$

The integral in Eq. (8.22) is very close to unity if $n_\alpha = n_\beta$ since, except the Coulomb interaction, the proton and neutron feel the same correlations. If isospin is conserved, and neglecting the influence of the Coulomb field upon $R_{n_\beta l_\beta j_\beta(P)}(r)$, the neutron and proton radial wave functions are the same and

$$\mathcal{M} = g_V \delta_{n_\alpha n_\beta} \delta_{l_\alpha l_\beta} \delta_{j_\alpha j_\beta} \tag{8.23}$$

In this case the initial and final nuclear states are isobaric analogous, that is they differ only by their isospin projections.

When isospin is conserved the Fermi matrix element must obey the isospin triangle condition $T_f - T_i$ ($\Delta T = 0$), and the Fermi operator can only connect isobaric analogue states. For β_- decay

$$T_- \mid \omega_i, J_i, M_i, T_i, T_{zi} >$$

$$= \sqrt{(T_i(T_i+1) - T_{zi}(T_{zi}-1)} \mid \omega_i, J_i, M_i, T_i, T_{zi} - 1 >,$$

and

$$B(F_-) = |< \omega_f, J_f, M_f, T_f, T_{zi} - 1 \mid T_- \mid \omega_i, J_i, M_i, T_i, T_{zi} >|^2$$

$$= [T_i(T_i+1) - T_{zi}(T_{zi}-1)]\delta_{\omega_f,\omega}\, \delta_{J_i,J_f}\, \delta_{M_i,M_f}\, \delta_{T_i,T_f}.$$

For neutron-rich nuclei ($N_i > Z_i$) we have $T_i = T_{zi}$ and thus

$$B(F_-)(N_i > Z_i) = 2T_{zi} = (N_i - Z_i)\, \delta_{\omega_f,\omega}\, \delta_{J_i,J_f}\, \delta_{M_i,M_f}\, \delta_{T_i,T_f},$$

For β_+ we have

$$B(F_+) = |< \omega_f, J_f, M_f, T_f, T_{zi}+1 \mid T_+ \mid \omega_i, J_i, M_i, T_i, T_{zi} >|^2$$
$$= [T_i(T_i+1) - T_{zi}(T_{zi}+1)]\delta_{\omega_f,\omega}\, \delta_{J_i,J_f}\, \delta_{M_i,M_f}\, \delta_{T_i,T_f}.$$

For proton-rich nuclei $(Z_i > N_i)$ we have $T_{zi} = -T_i$ and thus

$$B(F_+)(Z_i > N_i) = -2T_{zi} = (Z_i - N_i)\, \delta_{\omega_f,\omega}\, \delta_{J_i,J_f}\, \delta_{M_i,M_f}\, \delta_{T_i,T_f},$$

As such Fermi decay can only connect isobaric analogue states and it provides an exacting test of isospin conservation in the nucleus.

Gamow-Teller decay

The reduced single-particle matrix elements are given by

$$< k_a, p||\sigma t_-||k_b, n > = < k_a, n||\sigma t_+||k_b, p >= 2 < k_a||\vec{s}\,||k_b >,$$

where the matrix elements of $\vec{s}$ are given by

$$< k_a||\vec{s}\,||k_b > = < j_a||\vec{s}||j_b > \delta_{n_a,n_b}$$

$$= (-1)^{\ell_a+j_a+3/2}\sqrt{(2j_a+1)(2j_b+1)} \left\{ \begin{array}{ccc} 1/2 & 1/2 & 1 \\ j_b & j_a & \ell_a \end{array} \right\} < s||\vec{s}||s > \delta_{\ell_a,\ell_b}\delta_{n_a,n_b},$$

with

$$< s||\vec{s}||s > = \sqrt{3/2}\,,$$

The matrix elements of $\vec{s}$ has the selection rules δ_{ℓ_a,ℓ_b} and δ_{n_a,n_b}.

The sum rule

The sum rule for Fermi decay is obtained from the sum

$$\sum_f [B_{i,f}(F_-) - B_{i,f}(F_+)] = \sum_f \left[|< f \mid T_- \mid i >|^2 - |< f \mid T_+ \mid i >|^2\right]$$

The final states f in the T_- matrix element go with the $Z_f = Z_i + 1$ nucleus and those in the T_+ matrix element to with the $Z_f = Z_i - 1$ nucleus. One can explicitly sum over the final states to obtain

$$\sum_f [< i \mid T_+ \mid \underline{f >< f} \mid T_- \mid i > - < i \mid T_- \mid \underline{f >< f} \mid T_+ \mid i >]$$

$$= < i \mid T_+T_- - T_-T_+ \mid i > = < i \mid 2T_z \mid i > = (N_i - Z_i).$$

The sum rule for Gamow-Teller is obtained as follows.

$$\sum_{f,\mu} |< f \mid \sum_k \sigma_{k,\mu} t_{k-} \mid i >|^2 - \sum_{f,\mu} |< f \mid \sum_k \sigma_{k,\mu} t_{k+} \mid i >|^2$$

$$= \sum_{f,\mu} < i \mid \sum_k \sigma_{k,\mu} t_{k+} \mid \underline{f >< f} \mid \sum_{k'} \sigma_{k',\mu} t_{k'-} \mid i >$$

$$- \sum_{f,\mu} < i \mid \sum_k \sigma_{k,\mu} t_{k-} \mid \underline{f >< f} \mid \sum_k^{\prime} \sigma_{k',\mu} t_{k'+} \mid i >$$

$$= \sum_\mu \left[< i \mid \left(\sum_k \sigma_{k,\mu} t_{k+} \right) \left(\sum_{k'} \sigma_{k',\mu} t_{k'-} \right) - \left(\sum_k \sigma_{k,\mu} t_{k-} \right) \left(\sum_{k'} \sigma_{k',\mu} t_{k'+} \right) \mid i > \right]$$

$$= \sum_\mu < i \mid \sum_k \sigma_{k,\mu}^2 \, \underline{[t_{k+} t_{k-} - t_{k-} t_{k+}]} \mid i >= 3 < i \mid \sum_k [t_{k+} t_{k-} - t_{k-} t_{k+}] \mid i >$$

$$= 3 < i \mid T_+ T_- - T_- T_+ \mid i >= 3 < i \mid 2T_z \mid i >= 3(N_i - Z_i).$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1.$$

In particular for the β− decay of the neutron

$$B(F_-) = 1 \text{ and } B(GT_-) = 3.$$

For a pure Fermi transition ($S_\beta = 0, L_\beta = 0$), i.e. with the angular momentum restriction $\Delta J = 0$ and no change of parity, $|\bar{M}'_{fi}(F)|^2 = 1$. For a pure Gamow–Teller transition ($S_\beta = 1, L_\beta = 0$), i.e. with angular momentum change $\Delta J = 1$ and no change of parity, $|M'_{fi}(\mathrm{GT})|^2 = 3$. The latter result is obtained by summing over all possible spin $\frac{1}{2}$ orientations, using the spin $\frac{1}{2}$ eigenvectors (spinors), i.e.

$$\sum_{\mu, m_f} \left| \langle \chi_{1/2}^{m_f} | \hat{\sigma}_\mu | \chi_{1/2}^{m_i} \rangle \right|^2 = 3.$$

TABLE I. Experimental results for neutron lifetime.

Author(s), year [ref. no.]	τ_n (s) till 2007	τ_n (s), corrections and additions
Arzumanov *et al.*, 2009 [3]		881.5 ± 2.5
Ezhov *et al.*, 2007 [4]	878.2 ± 1.9	
Serebrov *et al.*, 2005 [1]	$878.5 \pm 0.7 \pm 0.3$	
Dewey *et al.*, 2003 [5]	$886.3 \pm 1.2 \pm 3.2$	
Arzumanov *et al.*, 2000 [6], Fomin and Serebrov, 2010 [7]	$885.4 \pm 0.9 \pm 0.4$	$879.9 \pm 0.9 \pm 2.4$
Pichlmaier *et al.*, 2000 [8]		881.0 ± 3
Byrne *et al.*, 1996 [9]	$889.2 \pm 3.0 \pm 3.8$	
Mampe *et al.*, 1993 [10]	882.6 ± 2.7	
Nesvizhevski *et al.*, 1992 [11]	$888.4 \pm 3.1 \pm 1.1$	Removed
Byrne *et al.*, 1990 [12]	$893.6 \pm 3.8 \pm 3.7$	
Mampe *et al.*, 1989 [13], Serebrov and Fomin, 2009 [14]	887.6 ± 3.0	881.6 ± 3.0
Kharitonov *et al.*, 1989 [15]	872 ± 8	
Kossakowski *et al.*, 1989 [16]	$878 \pm 27 \pm 14$	
Paul *et al.*, 1989 [17]	877 ± 10	
Spivac *et al.*, 1988 [18]	891 ± 9	
Last *et al.*, 1988 [19]	$876 \pm 10 \pm 19$	
Arnold *et al.*, 1987 [20]	870 ± 17	
Kosvintsev *et al.*, 1986 [21]	903 ± 13	
Byrne *et al.*, 1980 [22]	937 ± 18	
Bondarenko *et al.*, 1978 [23]	881 ± 8	
Christensen *et al.*, 1972 [24]	918 ± 14	

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Colloquium: The neutron lifetime

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14C-dating beta decay

$$^{14}C \rightarrow {}^{14}N + \beta^{-} + \overline{\nu}_e$$

The half-life of ^{14}C is 5,730±40 years. ^{14}C decays into nitrogen-14 through beta decay.

The presence of ^{14}C in organic materials is the basis of the radiocarbon dating method to date archaeological, geological, and hydrogeological samples. There are three naturally occurring isotopes of carbon on Earth: 99% of the carbon is carbon-12, 1% is carbon-13, and carbon-14 occurs in trace amounts, i.e. making up as much as 1 part per trillion of the carbon in the atmosphere.

The different isotopes of carbon do not differ appreciably in their chemical properties. This is used in chemical and biological research, in a technique called carbon labeling: ^{14}C atoms can used to trace chemical and biochemical reactions involving carbon atoms from any given organic compound.

Decay mode	Decay energy	Half-life
$^{11}C \rightarrow {}^{11}B + e^{+} + \nu_e$	1.98 MeV	20.4 min
$^{13}N \rightarrow {}^{13}C + e^{+} + \nu_e$	2.20 MeV	10.0 min
$^{14}O \rightarrow {}^{14}N + e^{+} + \nu_e$	5.14 MeV	1.18 min
$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_e$	2.75 MeV	2.04 min
$^{14}C \rightarrow {}^{14}N + e^{-} + \bar{\nu}_e$	0.156 MeV	2×10^{9} min

The involved single-particle orbits

$p_{1/2}$ and $p_{3/2}$

^{14}C: Two proton hole
^{14}N: One neutron hole+ one proton hole

Total isospin

T=0 (^{14}N) and 1 (^{14}C)

Total angular momentum

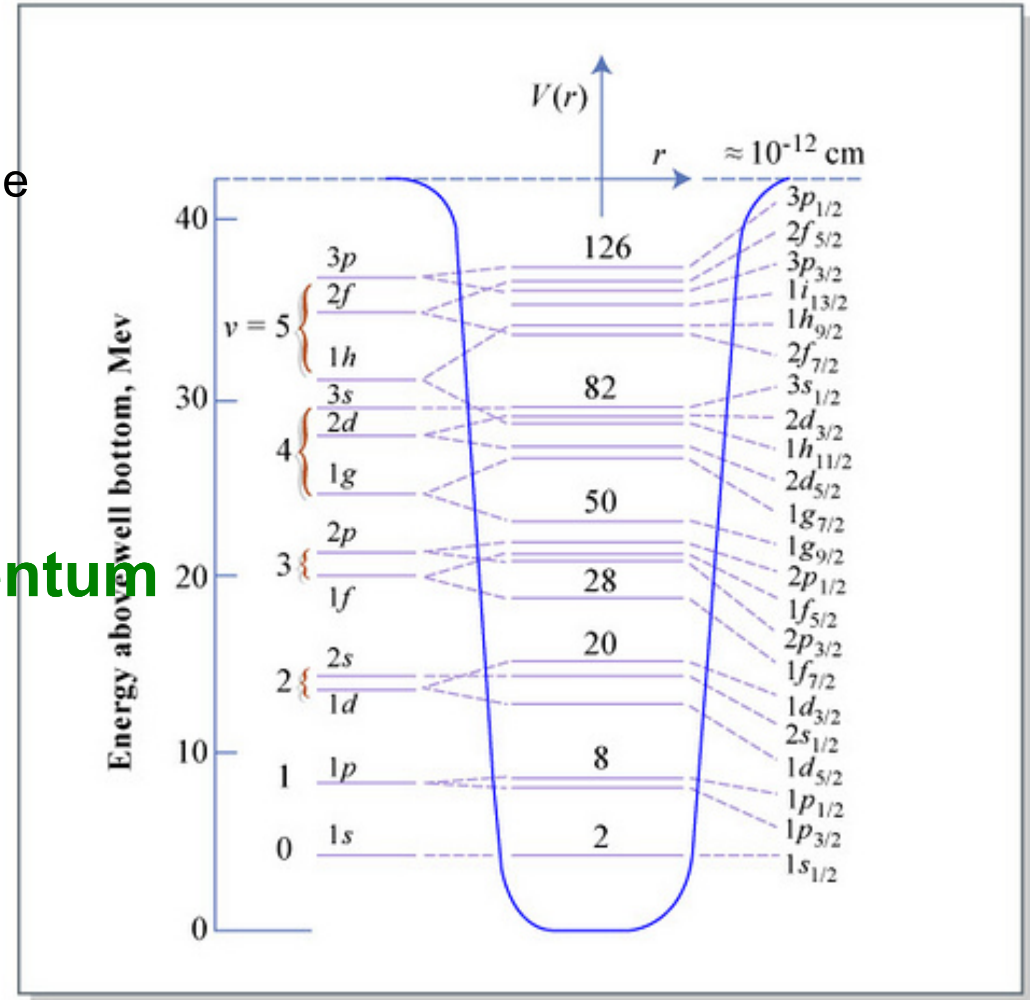
J=0(^{14}C),1 (^{14}N),2

Total orbital angular momentum

L=0,1,2

Total spin S

S=0, 1



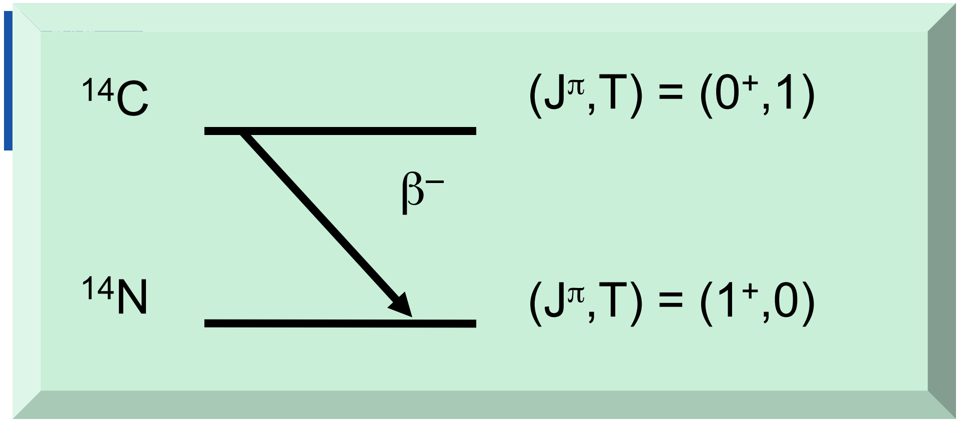
LS coupling

$$\Phi_{ll'SLJM} = \sum_{M_S M_L} (S M_S L M_L \mid SLJM) \chi_{SM_S} \Phi_{ll'LM_L}$$

$$\chi_{SM_S} = \sum_{m_s m_s'} (\tfrac{1}{2} m_s \tfrac{1}{2} m_s' \mid \tfrac{1}{2} \tfrac{1}{2} S M_S) \chi_{m_s}(1) \chi_{m_s'}(2)$$

$$\Phi_{ll'LM_L}(1,2) = \frac{1}{\sqrt{2}} \sum (l m_l l' m_l' \mid ll' L M_L)$$

$$\times [\phi_{lm_l}(\mathbf{r}_1)\phi_{l'm_l'}(\mathbf{r}_2) \pm \phi_{lm_l}(\mathbf{r}_2)\phi_{l'm_l'}(\mathbf{r}_1)]$$



$$\psi_i = x\left|{}^1S_0\right\rangle + y\left|{}^3P_0\right\rangle$$

$$\psi_f = a\left|{}^3S_1\right\rangle + b\left|{}^1P_1\right\rangle + c\left|{}^3D_1\right\rangle$$

$$M_{GT} = \sum_k \left\langle \psi_f \left\| \tau_+(k)\sigma(k) \right\| \psi_i \right\rangle \approx \pm\, 2\times10^{-3}$$

$$M_{GT} = -\sqrt{6}\left(xa - yb/\sqrt{3}\right)$$

x and y have the same sign (pairing collectivity) and similar strength

a and b should also have the same sign

14N ground state is dominated by the D component

a is induced by the tensor force

Calculation[a]	Ground state $(J^\pi, T) = (1^+, 0)$		
	C_S	C_P	C_D
Elliott	+0.077	+0.179	+0.981
Visscher and Ferrell	+0.173	+0.355	+0.920
Cohen and Kurath I	+0.088	+0.261	+0.962
Cohen and Kurath II	+0.136	+0.240	+0.962

jj coupling

$$\Psi_{ll'jj'JM}(1,2) = \frac{1}{\sqrt{2}} \sum_{mm'} (jmj'm' \mid jj'JM)$$

$$\times [\psi_{jm}(1)\psi_{j'm'}(2) \pm \psi_{jm}(2)\psi_{j'm'}(1)]$$

The transformation

$$\Psi_{ll'jj'JM} = \sum_{S,L} \langle \tfrac{1}{2}l(j)\tfrac{1}{2}l'(j')J \mid \tfrac{1}{2}\tfrac{1}{2}(S)ll'(L)J \rangle \Psi_{ll'SLJM}$$

$$|\psi_i\rangle = \kappa|0p_{1/2}^{-2}\rangle + \eta|0p_{3/2}^{-2}\rangle,$$

$$|\psi_f\rangle = a|0p_{1/2}^{-2}\rangle + b|0p_{3/2}^{-1}0p_{1/2}^{-1}\rangle + c|0p_{3/2}^{-2}\rangle,$$

$$M(\mathrm{GT}) = \langle\psi_f||\sigma\tau||\psi_i\rangle = \sqrt{\frac{2}{3}}\left[\kappa(a+2b) + \eta(\sqrt{2}b - \sqrt{5}c)\right].$$

The ground state of ^{14}C is dominated by the configuration of $|0p_{1/2}^{-2}\rangle$ due to the large spin-orbit splitting between orbits $0p_{1/2}^{-1}$ and $0p_{3/2}^{-1}$. This is supported by calculations

b~-0.5a

Interaction	η	κ	c	b	a
Empirical					
CK [19]	0.38	0.92	-0.027	-0.31	0.95
VWG [20]	0.36	0.93	-0.063	-0.27	0.96
WBT [21]	0.31	0.95	0.033	-0.43	0.90
WBP [21]	0.30	0.95	0.014	-0.41	0.91
JT [5]	0.09	0.99	0.20	-0.41	0.89
Zamick [7]	0.22	0.98	0.014	-0.40	0.92
VF [27]	0.25	0.97	0.12	-0.36	0.97
Realistic					
CD-Bonn(K)	0.40	0.92	0.20	-0.77	0.61
$N^3LO(K)$	0.39	0.92	0.15	-0.71	0.69
CD-Bonn(G)	0.39	0.92	0.14	-0.70	0.70
$N^3LO(G)$	0.38	0.93	0.11	-0.65	0.75
Chiral [11]	0.40	0.92	0.14	-0.68	0.72

Table 4. The central, spin-orbit (SO) and tensor components of the matrix elements $\langle ij|V|kl\rangle^{JT}$ of empirical and realistic interactions.

	VWG			WBT			CD-Bonn(K)			N^3LO(G)		
$ijkl$	Central	SO	Tensor	Central	SO	Tensor	Central	SO	Tensor	Central	SO	Tensor
			$J^\pi = 0^+, T = 1$									
1111	-1.66	0.58	0.92	-1.33	054	-0.42	-2.14	0.37	1.00	-2.02	0.38	1.04
3333	-4.43	0.29	0.46	-3.96	0.33	-0.21	-4.93	0.19	0.50	-4.56	0.17	0.52
1133	-3.93	-0.41	-0.65	-3.72	-0.42	0.29	-3.94	-0.27	-0.71	-3.58	-0.26	-0.73
			$J^\pi = 1^+, T = 0$									
1111	-4.27	0.41	-0.098	-4.49	1.11	-0.075	-4.64	-0.10	1.17	-4.65	-0.15	1.15
1113	1.08	-0.21	0.83	0.43	-0.0019	1.38	1.17	0.26	0.39	1.29	0.30	0.50
1313	-5.85	0.10	-0.80	-5.85	0.35	-1.36	-7.70	0.13	-0.69	-7.46	0.15	-0.78
1133	2.72	-0.13	-0.46	1.55	-0.046	-0.82	3.80	0.07	-0.99	3.76	0.18	-1.03
1333	3.72	0.065	-0.016	2.41	0.045	-0.012	5.73	0.01	0.19	5.49	0.02	0.18
3333	-2.96	0.041	0.30	-4.11	-0.58	0.53	-3.49	-0.32	0.51	-3.34	-0.36	0.54

Table 5. Wave functions of ^{14}C and ^{14}N calculated with the central force and central and spin-orbit force components of empirical and realistic effective interactions.

Interaction	η	κ	c	b	a
			Central force only		
CK [19]	0.33	0.94	-0.15	-0.085	0.99
VWG [20]	0.29	0.96	-0.14	-0.086	0.99
WBT [21]	0.30	0.95	-0.11	-0.033	0.99
CD-Bonn(K)	0.33	0.94	-0.30	0.14	0.94
N^3LO(K)	0.32	0.95	-0.20	-0.017	0.98
CD-Bonn(G)	0.32	0.95	-0.27	0.0080	0.96
N^3LO(G)	0.30	0.95	-0.25	0.037	0.97
			Central plus spin-orbit		
CK [19]	0.34	0.94	-0.15	-0.085	0.99
VWG [20]	0.32	0.95	-0.15	-0.053	0.99
WBT [21]	0.33	0.94	-0.12	-0.030	0.99
CD-Bonn(K)	0.35	0.94	-0.27	0.04	0.96
N^3LO(K)	0.34	0.94	-0.18	-0.11	0.98
CD-Bonn(G)	0.34	0.94	-0.25	-0.0071	0.97
N^3LO(G)	0.32	0.95	-0.23	-0.054	0.97

$$\Psi_{ll'jj'JM} = \sum_{S,L} \langle \tfrac{1}{2}l(j)\tfrac{1}{2}l'(j')J \mid \tfrac{1}{2}\tfrac{1}{2}(S)ll'(L)J \rangle \Psi_{ll'SLJM}$$

TABLE 9.1 *Transformation coefficients between jj-coupling and LS-coupling schemes.* $\Sigma = l + l' + 1$, $\Delta = l - l'$. *For* $l = l'$, $n = n'$, *the second row should be multiplied by* $\sqrt{2}$ *and the third row ignored.*

LS-coupling / jj-coupling	$^3(J+1)$ $S=1, L=J+1$	^1J $S=0, L=J$	$^3(J-1)$ $S=1, L=J-1$	^3J $S=1, L=J$
$j = l + \frac{1}{2}$ $j' = l' + \frac{1}{2}$	$-\sqrt{\frac{((J+1)^2-\Delta^2)(\Sigma-J)(\Sigma-J-1)}{2(\Sigma^2-\Delta^2)(J+1)(2J+1)}}$	$\sqrt{\frac{(\Sigma-J)(\Sigma+J+1)}{2(\Sigma^2-\Delta^2)}}$	$\sqrt{\frac{(J^2-\Delta^2)(\Sigma+J)(\Sigma+J+1)}{2(\Sigma^2-\Delta^2)J(2J+1)}}$	$-\Delta\sqrt{\frac{(\Sigma-J)(\Sigma+J+1)}{2(\Sigma^2-\Delta^2)J(J+1)}}$
$j = l + \frac{1}{2}$ $j' = l' - \frac{1}{2}$	$\sqrt{\frac{\left(\Sigma^2-(J+1)^2\right)(J-\Delta)(J-\Delta+1)}{2(\Sigma^2-\Delta^2)(J+1)(2J+1)}}$	$-\sqrt{\frac{(J-\Delta)(J+\Delta+1)}{2(\Sigma^2-\Delta^2)}}$	$\sqrt{\frac{\left(\Sigma^2-J^2\right)(J+\Delta)(J+\Delta+1)}{2(\Sigma^2-\Delta^2)J(2J+1)}}$	$\Sigma\sqrt{\frac{(J-\Delta)(J+\Delta+1)}{2(\Sigma^2-\Delta^2)J(J+1)}}$
$j = l - \frac{1}{2}$ $j' = l' + \frac{1}{2}$	$-\sqrt{\frac{\left(\Sigma^2-(J+1)^2\right)(J+\Delta)(J+\Delta+1)}{2(\Sigma^2-\Delta^2)(J+1)(2J+1)}}$	$\sqrt{\frac{(J+\Delta)(J-\Delta+1)}{2(\Sigma^2-\Delta^2)}}$	$-\sqrt{\frac{\left(\Sigma^2-J^2\right)(J-\Delta)(J-\Delta+1)}{2(\Sigma^2-\Delta^2)J(2J+1)}}$	$\Sigma\sqrt{\frac{(J+\Delta)(J-\Delta+1)}{2(\Sigma^2-\Delta^2)J(J+1)}}$
$j = l - \frac{1}{2}$ $j' = l' - \frac{1}{2}$	$\sqrt{\frac{((J+1)^2-\Delta^2)(\Sigma+J)(\Sigma+J+1)}{2(\Sigma^2-\Delta^2)(J+1)(2J+1)}}$	$\sqrt{\frac{(\Sigma+J)(\Sigma-J-1)}{2(\Sigma^2-\Delta^2)}}$	$-\sqrt{\frac{(J^2-\Delta^2)(\Sigma-J)(\Sigma-J-1)}{2(\Sigma^2-\Delta^2)J(2J+1)}}$	$\Delta\sqrt{\frac{(\Sigma+J)(\Sigma-J-1)}{2(\Sigma^2-\Delta^2)J(J+1)}}$

$$|p_{3/2}^2 J = 1\rangle = -\frac{1}{3}\sqrt{\frac{2}{3}}\,{}^3D_1 + \frac{\sqrt{5}}{3}\,{}^1P_1 + \frac{1}{3}\sqrt{\frac{10}{3}}\,{}^3S_1$$

$$|p_{3/2}p_{1/2}T = 0, J = 1\rangle = \frac{1}{3}\sqrt{\frac{5}{3}}\,{}^3D_1 - \frac{\sqrt{2}}{3}\,{}^1P_1 + \frac{4}{3\sqrt{3}}\,{}^3S_1$$

The transformation between wave functions in LS and jj coupling schemes is known in analytic forms in terms of 6j and 9j symbols.

$$\begin{pmatrix} |^1S_0\rangle \\ |^3P_0\rangle \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} |0p_{1/2}^{-2}\rangle \\ |0p_{3/2}^{-2}\rangle \end{pmatrix},$$

and

$$\begin{pmatrix} |^3S_1\rangle \\ |^1P_1\rangle \\ |^3D_1\rangle \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & -4 & \sqrt{10} \\ \sqrt{6} & \sqrt{6} & \sqrt{15} \\ \sqrt{20} & -\sqrt{5} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} |0p_{1/2}^{-2}\rangle \\ |0p_{3/2}^{-1}0p_{1/2}^{-1}\rangle \\ |0p_{3/2}^{-2}\rangle \end{pmatrix}$$

THANK YOU!