KTH ROYAL INSTITUTE
OF TECHNOLOGY

Two-particle states
April 19, 2017

-Pairing (seniority) and neutron-proton coupling

Slater determinant

For identical nucleons, the simple product wave function is not appropriate, since it must describe indistinguishable particles and account for the Pauli exclusion principle

$$\Psi(1,2,\ldots,A) = \prod_{i=1}^{A} \phi(i)$$

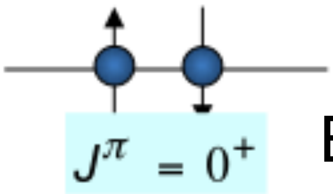
$$\Psi_\alpha(1,2,\ldots A) \equiv \Psi_{\alpha_1,\alpha_2,\ldots,\alpha_A}(1,2,\ldots A) = \frac{1}{\sqrt{A!}} \sum_P (-1)^P \prod_{i=1}^{i=A} \psi_{\alpha_i}(\vec{r}_i),$$

A normalized, antisymmetric A-particle wave function is defined by the Slater determinant as

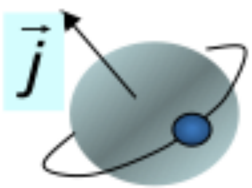
$$\frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{a_1}(1) & \phi_{a_1}(2) & \cdots & \phi_{a_1}(A) \\ \phi_{a_2}(1) & \phi_{a_2}(2) & \cdots & \phi_{a_2}(A) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \phi_{a_A}(1) & \cdots & \cdots & \phi_{a_A}(A) \end{vmatrix}$$

Seniority symmetry

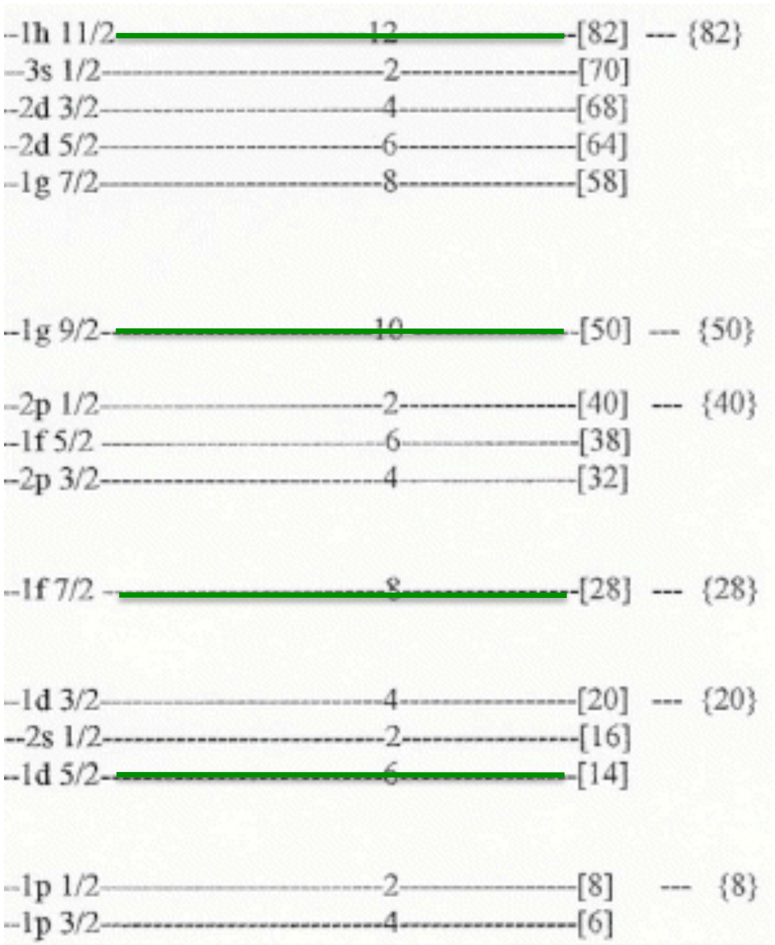
1943 Racah
1949 Goeppert-Mayer



Even-even



Odd-A

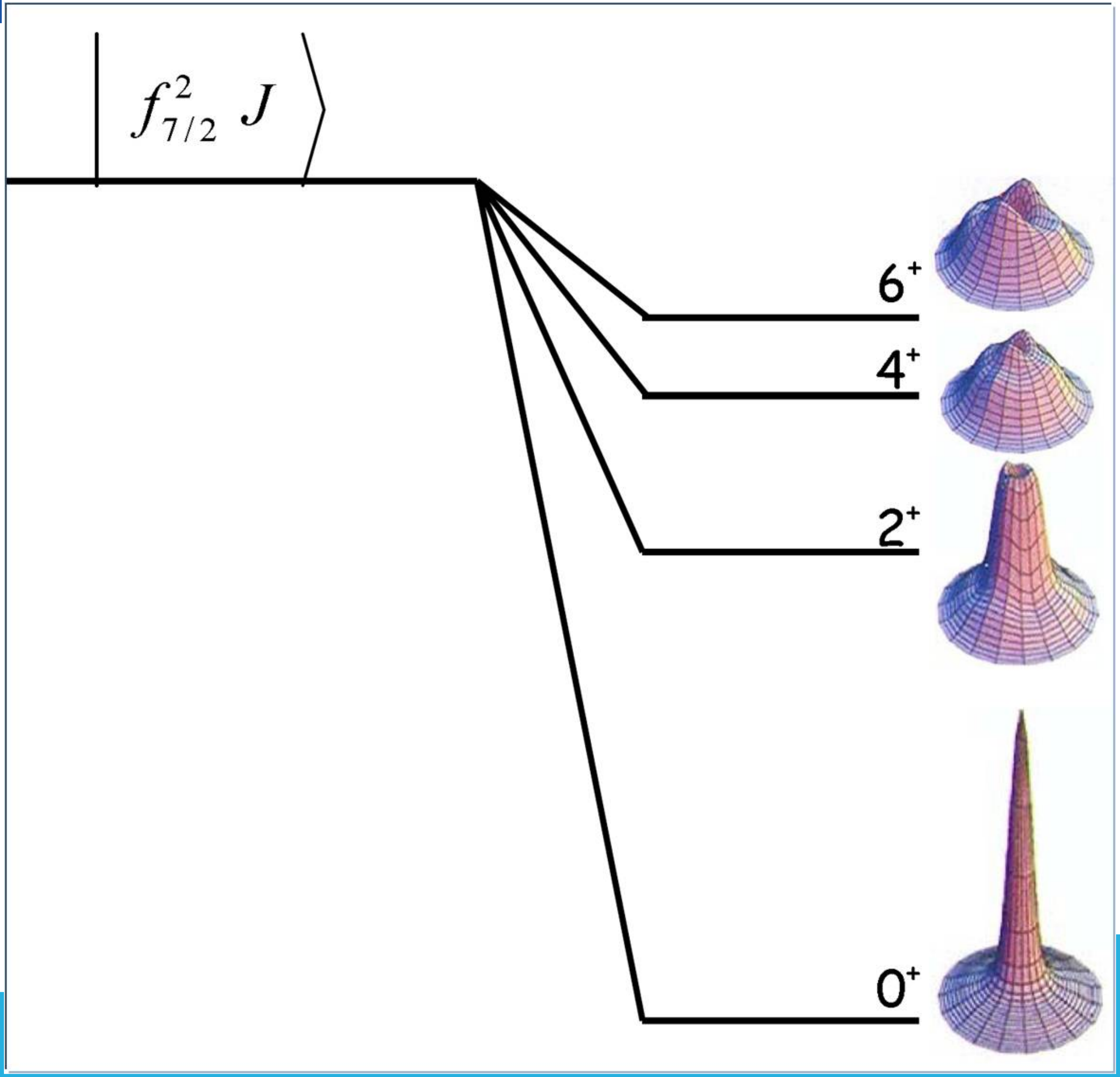


Mean field theories

- Single-particle model
  HO, WS…
- Hartree-Fock (density functional) approaches
  Skyrme force, boson exchange potentials
- Shell model
  Monopole

Two particles in a single j shell

The *J=0 pairing interaction* is the dominant component of the nuclear interaction.



Sum of angular momenta

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2$$

$$[\hat{\boldsymbol{L}}^2, \hat{L}_i] = 0 \quad (i = x, y, z)$$

$$[\hat{L}_x, \hat{L}_y] = \mathrm{i}\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = \mathrm{i}\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = \mathrm{i}\hbar \hat{L}_y$$

$$[\hat{\boldsymbol{L}}_1, \hat{\boldsymbol{L}}_2] = 0, \quad [\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}_1] = [\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}_2] = 0$$

$$\begin{aligned}
\hat{\boldsymbol{L}}_1^2|l_1 m_1\rangle &= \hbar^2 l_1(l_1+1)|l_1 m_1\rangle \quad ; \quad \hat{L}_{1z}|l_1 m_1\rangle = \hbar m_1 |l_1 m_1\rangle \\
\hat{\boldsymbol{L}}_2^2|l_2 m_2\rangle &= \hbar^2 l_2(l_2+1)|l_2 m_2\rangle \quad ; \quad \hat{L}_{2z}|l_2 m_2\rangle = \hbar m_2 |l_2 m_2\rangle \\
\hat{\boldsymbol{L}}^2|l m\rangle &= \hbar^2 l(l+1)|l m\rangle \quad ; \quad \hat{L}_{z}|l m\rangle = \hbar m |l m\rangle
\end{aligned}$$

$$|l_1 - l_2| \leqslant l \leqslant l_1 + l_2, \quad m = m_1 + m_2$$

$$-l_i \leqslant m_i \leqslant l_i, \quad -l \leqslant m \leqslant l$$

If the operators A and B commute, then they have common eigenvectors (eigenvalues can be different

$$\hat{A}|\alpha\rangle = a|\alpha\rangle \quad \text{and} \quad \hat{B}|\beta\rangle = b|\beta\rangle$$

$$\hat{A}|\alpha\beta\rangle = a|\alpha\beta\rangle, \quad \hat{B}|\alpha\beta\rangle = b|\alpha\beta\rangle$$

In our angular momenta case the standard choice is

$$\hat{\boldsymbol{L}}_1^2,\ \hat{L}_{1z},\ \hat{\boldsymbol{L}}_2^2,\ \hat{L}_{2z} \text{ or } \hat{\boldsymbol{L}}_1^2,\ \hat{\boldsymbol{L}}_2^2,\ \hat{\boldsymbol{L}}^2,\ \hat{L}_z$$

Corresponding to the representations

$$|l_1 m_1 l_2 m_2\rangle \quad \text{or} \quad |l_1 l_2 l m\rangle$$

The projectors are

$$\sum_{l_1 m_1 l_2 m_2} |l_1 m_1 l_2 m_2\rangle\langle l_1 m_1 l_2 m_2| = \hat{I} \quad \text{or} \quad \sum_{l_1 l_2 l m} |l_1 l_2 l m\rangle\langle l_1 l_2 l m| = \hat{I}$$

One can then write

$$|l_1 m_1 l_2 m_2\rangle = \sum_{lm} |l_1 l_2 l m\rangle\langle l_1 l_2 l m | l_1 m_1 l_2 m_2\rangle$$

$\langle l_1 m_1 l_2 m_2 | l m\rangle = \langle l_1 l_2 l m | l_1 m_1 l_2 m_2\rangle$ is real and is called Clebsch-Gordan

$$|l_1 m_1 l_2 m_2\rangle = \sum_{lm} \langle l_1 m_1 l_2 m_2 | l m\rangle |l_1 l_2 l m\rangle$$

In the same fashion

$$|l_1 l_2 l m\rangle = \sum_{m_1 m_2} \langle l_1 m_1 l_2 m_2 | l m\rangle |l_1 m_1 l_2 m_2\rangle$$

$$\langle \boldsymbol{r}|p\rangle = R_{n_p l_p j_p}(r)\left[Y_{l_p}(\hat{r})\chi_{1/2}\right]_{j_p m_p}$$

Uncoupled scheme

$$\Psi_a(pq;\boldsymbol{r}_1\boldsymbol{r}_2) = \frac{1}{\sqrt{2}}\Big[R_p(r_1)\left[Y_{l_p}(\hat{r}_1)\chi_{1/2}\right]_p R_q(r_2)\left[Y_{l_q}(\hat{r}_2)\chi_{1/2}\right]_q - R_p(r_2)\left[Y_{l_p}(\hat{r}_2)\chi_{1/2}\right]_p R_q(r_1)\left[Y_{l_q}(\hat{r}_1)\chi_{1/2}\right]_q\Big]$$

jj coupling scheme

We have also seen that one can choose another coupling scheme, in which the two particles carry total angular momentum $\boldsymbol{J} = \boldsymbol{j}_p + \boldsymbol{j}_q$ with projection $M = m_p + m_q$, such that $|j_p - j_q| \leqslant J \leqslant j_p + j_q$ and $-J \leqslant M \leqslant J$. This is called *coupled*-scheme. In this scheme the two-particle wave function reads,

$$\Psi_a(pq, JM;\boldsymbol{r}_1\boldsymbol{r}_2) = \mathcal{N}\Big[R_p(r_1)R_q(r_2)\left[\left[Y_{l_p}(\hat{r}_1)\chi_{1/2}\right]_{j_p}\left[Y_{l_q}(\hat{r}_2)\chi_{1/2}\right]_{j_q}\right]_{JM} - R_p(r_2)R_q(r_1)\left[\left[Y_{l_p}(\hat{r}_2)\chi_{1/2}\right]_{j_p}\left[Y_{l_q}(\hat{r}_1)\chi_{1/2}\right]_{j_q}\right]_{JM}\Big] \quad (3)$$

The quantum numbers associated to this wave function are $\{n_p n_q l_p l_q j_p j_q J M\}$.

$$\big\{|\alpha\rangle = |pq;JM\rangle\big\}$$

In an coupled scheme for two particles in the shells p and q one has $\boldsymbol{J} = \boldsymbol{j}_1 + \boldsymbol{j}_2$ and $\boldsymbol{M} = \boldsymbol{m}_1 + \boldsymbol{m}_2$

$$\langle 12|j_p j_q; JM\rangle = \sum_{m_p m_q} \langle j_p m_p j_q m_q|JM\rangle\langle 12|j_p m_p j_q m_q\rangle \tag{4.6}$$

Antisymmetry

$$\begin{aligned}\langle 12|j_p j_q; JM\rangle_a &= N\big(\langle 12|j_p j_q; JM\rangle - \langle 21|j_p j_q; JM\rangle\big) \\ &= N\sum_{m_p m_q}\langle j_p m_p j_q m_q|JM\rangle\big(\langle 12|j_p m_p j_q m_q\rangle - \langle 21|j_p m_p j_q m_q\rangle\big)\end{aligned} \tag{4.7}$$

But

$$\langle 21|j_p m_p j_q m_q\rangle = \langle 2|j_p m_p\rangle\langle 1|j_q m_q\rangle = \langle 12|j_q m_q j_p m_p\rangle \tag{4.8}$$

one gets

$$\langle 12|j_p j_q; JM\rangle_a = N\big(\langle 12|j_p j_q; JM\rangle - (-1)^{j_p+j_q-J}\langle 12|j_q j_p; JM\rangle\big) \tag{4.9}$$

If $j_p = j_q$, then $(-1)^{j_p+j_q} = -1$ and

$$\langle 12|j_p^2; JM\rangle_a = N\big(1 + (-1)^J\big)\langle 12|j_p^2; JM\rangle \tag{4.10}$$

J must be even and $N = 1/2$ (since $\langle j_p^2; JM|j_p^2; JM\rangle = 1$). Otherwise $N = 1/\sqrt{2}$.

Parity of n-particle states

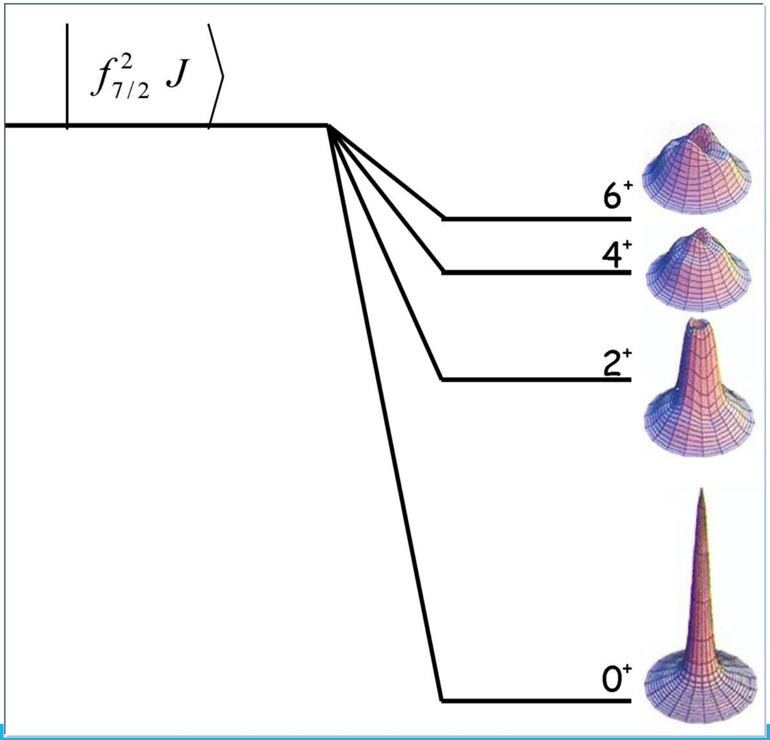
$$\Psi_a(pq, JM; \boldsymbol{r}_1\boldsymbol{r}_2) = \mathcal{N}\Big[R_p(r_1)R_q(r_2)\left[\left[Y_{l_p}(\hat{r}_1)\chi_{1/2}\right]_{j_p}\left[Y_{l_q}(\hat{r}_2)\chi_{1/2}\right]_{j_q}\right]_{JM} - R_p(r_2)R_q(r_1)(-1)^{j_p+j_q-J}\left[\left[Y_{l_q}(\hat{r}_1)\chi_{1/2}\right]_{j_q}\left[Y_{l_p}(\hat{r}_2)\chi_{1/2}\right]_{j_p}\right]_{JM}\Big]$$

$$(\mathbf{r_1}, \mathbf{r_2}) \rightarrow (-\mathbf{r_1}, -\mathbf{r_2})$$

$$(-1)^{l_p+l_q}\left[Y_{l_p}(\hat{r}_1)\chi_{1/2}\right]_{j_pm_p}\left[Y_{l_q}(\hat{r}_2)\chi_{1/2}\right]_{j_qm_q}$$

Spin of n-particle states in a single-j shell

$$\Psi_a(pq, JM; \boldsymbol{r}_1\boldsymbol{r}_2) = \mathcal{N}\Big[R_p(r_1)R_q(r_2)\left[\left[Y_{l_p}(\hat{r}_1)\chi_{1/2}\right]_{j_p}\left[Y_{l_q}(\hat{r}_2)\chi_{1/2}\right]_{j_q}\right]_{JM}$$
$$- R_p(r_2)R_q(r_1)(-1)^{j_p+j_q-J}\left[\left[Y_{l_q}(\hat{r}_1)\chi_{1/2}\right]_{j_q}\left[Y_{l_p}(\hat{r}_2)\chi_{1/2}\right]_{j_p}\right]_{JM}\Big]$$



Isospin

The isospin is an *internal quantum number* used in the classification of elementary particles

Isospin was introduced by Werner Heisenberg in 1932 to explain the fact that the strength of the strong interaction is almost the same between two protons or two neutrons as between a proton and a neutron

isospin is not a perfect symmetry but a good approximate symmetry of the strong interaction

Two-nucleon holes

PRL 97, 152501 (2006)

Protons and neutrons not only possess almost the same mass *(M_n/M_p = 1.0014)*, but they also show a far reaching symmetry with regard to the nuclear interaction.

Analysis of the results obtained experimentally from scattering protons by protons and neutrons by protons show that the nuclear forces for the pp and np system are equal to-within a few percent.

The symmetric properties of mirror nuclei and other sets of isobars provides further pertinent information. The energy spectra of mirror nuclei (i.e. nuclei which transform into each other by an interchange of protons and neutrons) are very much alike. The differences in excitation energies are very small and are mainly due to Coulomb forces.

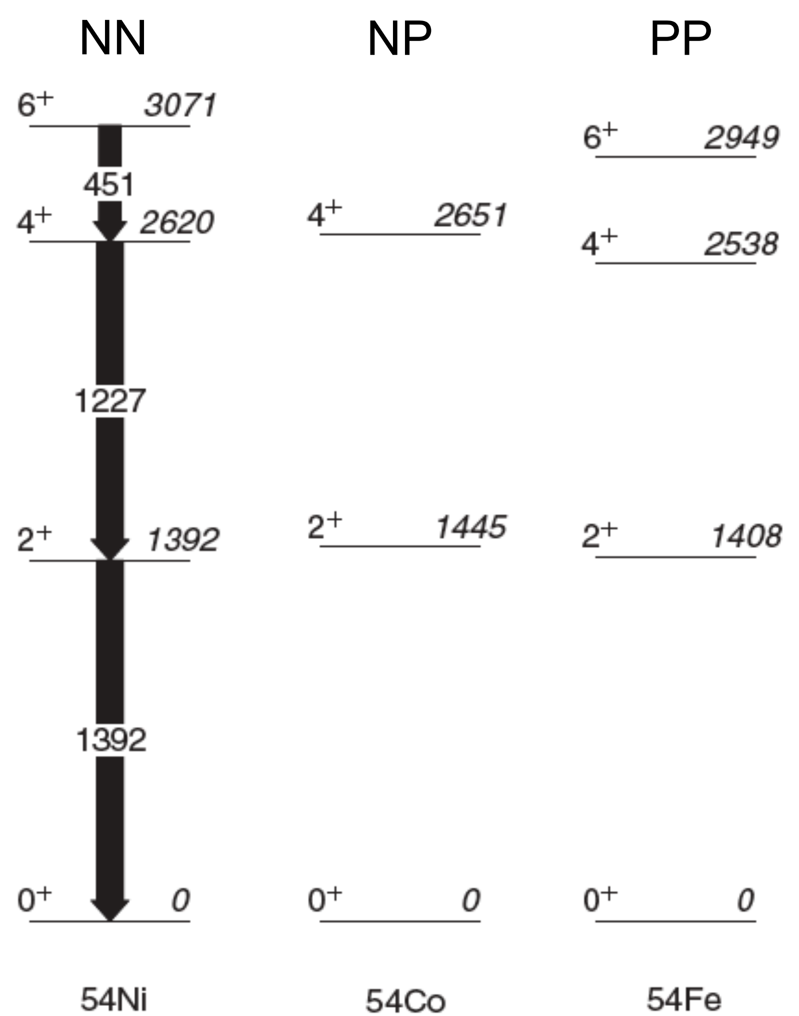
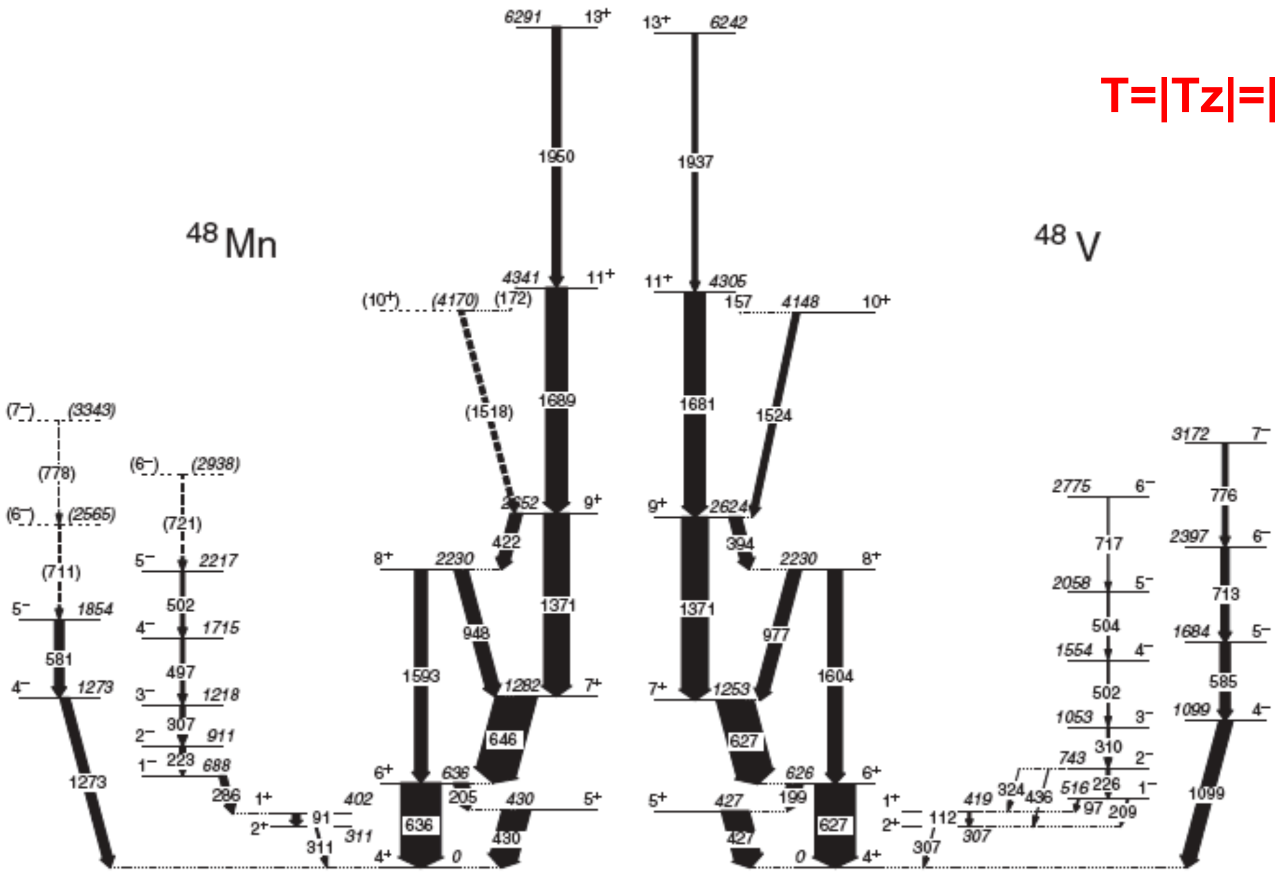


FIG. 2. Level scheme of ^{54}Ni, as deduced in this work, compared to those of ^{54}Co and ^{54}Fe.

it turns out that the influence of the Coulomb field can still be treated in many cases as a perturbation in the energy, leaving the total isospin quantum number ***T*** virtually unimpaired as a state label. At present one has obtained a large body of data on isomultiplets corroborating the assumption of charge independence of the nuclear forces.

T=|Tz|=|N-Z|/2=3/2



PRL **97, 132501 (2006)**

In 1932, W. Heisenberg proposed that the neutron and proton could be considered as two different charge states of one and the same particle by introducing an isospin variable *t for which the projection* t_z *can as*sume two values: $t_z=-1/2$ *to label a proton and* $t_z=1/2$ *to label a neutron.*

The isospin formalism can be developed in complete analogy with the description of intrinsic spin in terms of Pauli spinors with the two possibilities of spin up and spin down. It **is given by the invariance of the Hamiltonian of the strong interactions under the action of the Lie group SU(2). The neutron and the proton are assigned to the doublet (of the fundamental representation) of SU(2):**

A neutron wave function is represented by the isospinor

$$\phi_n(r) = \begin{pmatrix} \phi(r) \\ 0 \end{pmatrix} = \phi(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad t_z \phi_n(r) = +\tfrac{1}{2}\phi_n(r)$$

The proton wave function

$$\phi_p(r) = \begin{pmatrix} 0 \\ \phi(r) \end{pmatrix} = \phi(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad t_z \phi_p(r) = -\tfrac{1}{2}\phi_p(r)$$

The single-particle wave function can then be labeled by the spatial-spin quantum numbers $|nlsjm\rangle$ (where $s = 1/2$) and the isospin quantum numbers $|tt_z\rangle$ ($t = 1/2$). The corresponding wave function is,

$$\langle \mathbf{r}|nlsjmtt_z\rangle = R_{nlj}(r)\left[Y_l(\hat{r})\chi_s\right]_{jm}\tau_{tt_z} \tag{18}$$

where τ_{tt_z} is the isospin state.

Introducing the analogues of the Pauli matrices into isospin space

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

one defines the isospin vector operator as

$$\boldsymbol{t} = \tfrac{1}{2}\boldsymbol{\tau}.$$

it follows that the components t_x, t_y and t_z obey the commutation relations of an angular momentum

$$[t_x, t_y] = i\, t_z$$

$$[\boldsymbol{t}^2, t_z] = 0$$

and that the eigenvalues of $\boldsymbol{t}^2$ are given by

$$\boldsymbol{t}^2 \Rightarrow t(t+1).$$

It should be stressed that *the isospin bears no relation to ordinary space.*

In analogy with angular momentum one can define raising and lowering operators as

$$t_+ = \frac{1}{2}\tau_+ = \frac{1}{2}(\tau_x + i\tau_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$t_- = \frac{1}{2}\tau_- = \frac{1}{2}(\tau_x - i\tau_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The operators $\tau_\pm$ that transform a proton state into a neutron state and *vice versa*

Direct application of the explicit matrix representations to the isospinors yields the relations

$$t_+\phi_p = \phi_n, \qquad t_+\phi_n = 0, \qquad t_-\phi_p = 0, \qquad t_-\phi_n = \phi_p$$

In a composite system of two or more nucleons the (commuting) individual isospins may be coupled to a total isospin

$$T = \sum_{k=1}^{A} t(k) ,$$

Again in analogy with the composition of ordinary spins, one finds that the vector operator *T obeys the angular* momentum commutation relations and hence the following eigenvalues are

$$T^2 \Rightarrow T(T+1) ,$$

For a given eigenvalueT the state is seen to be *(2T + 1)*-fold degenerate

$$T_z \Rightarrow -T, -T+1, \ldots, T-1, T .$$

Tz is related to the numbers of neutrons and protons by

$$T_z = \tfrac{1}{2}(N - Z)$$

Isospin of the two-nucleon system

For the isospin part of the wave functions we have the uncoupled basis consisting of the four states (the indices 1 and 2 refer to the two particles)

$$|p_1\rangle|p_2\rangle \quad |p_1\rangle|n_2\rangle \quad |n_1\rangle|p_2\rangle \quad |n_1\rangle|n_2\rangle$$

In the coupled basis we have

	$T_z =$	$+1$	$|p_1\rangle|p_2\rangle$
$T=1$	$T_z =$	0	$\frac{1}{\sqrt{2}}(|p_1\rangle|n_2\rangle + |n_1\rangle|p_2\rangle)$
	$T_z =$	-1	$|n_1\rangle|n_2\rangle$
$T=0$	$T_z =$	0	$\frac{1}{\sqrt{2}}(|p_1\rangle|n_2\rangle - |n_1\rangle|p_2\rangle)$

The states with total isospin T=1 are *symmetric* under exchange of the two particles, whereas the state with isospin T=0 is *antisymmetric*

$$\langle 12|j_pj_q;JM\rangle_a = N\big(\langle 12|j_pj_q;JM\rangle - (-1)^{j_p+j_q-J}\langle 12|j_qj_p;JM\rangle\big) \tag{4.9}$$

If $j_p = j_q$, then $(-1)^{j_p+j_q} = -1$ and

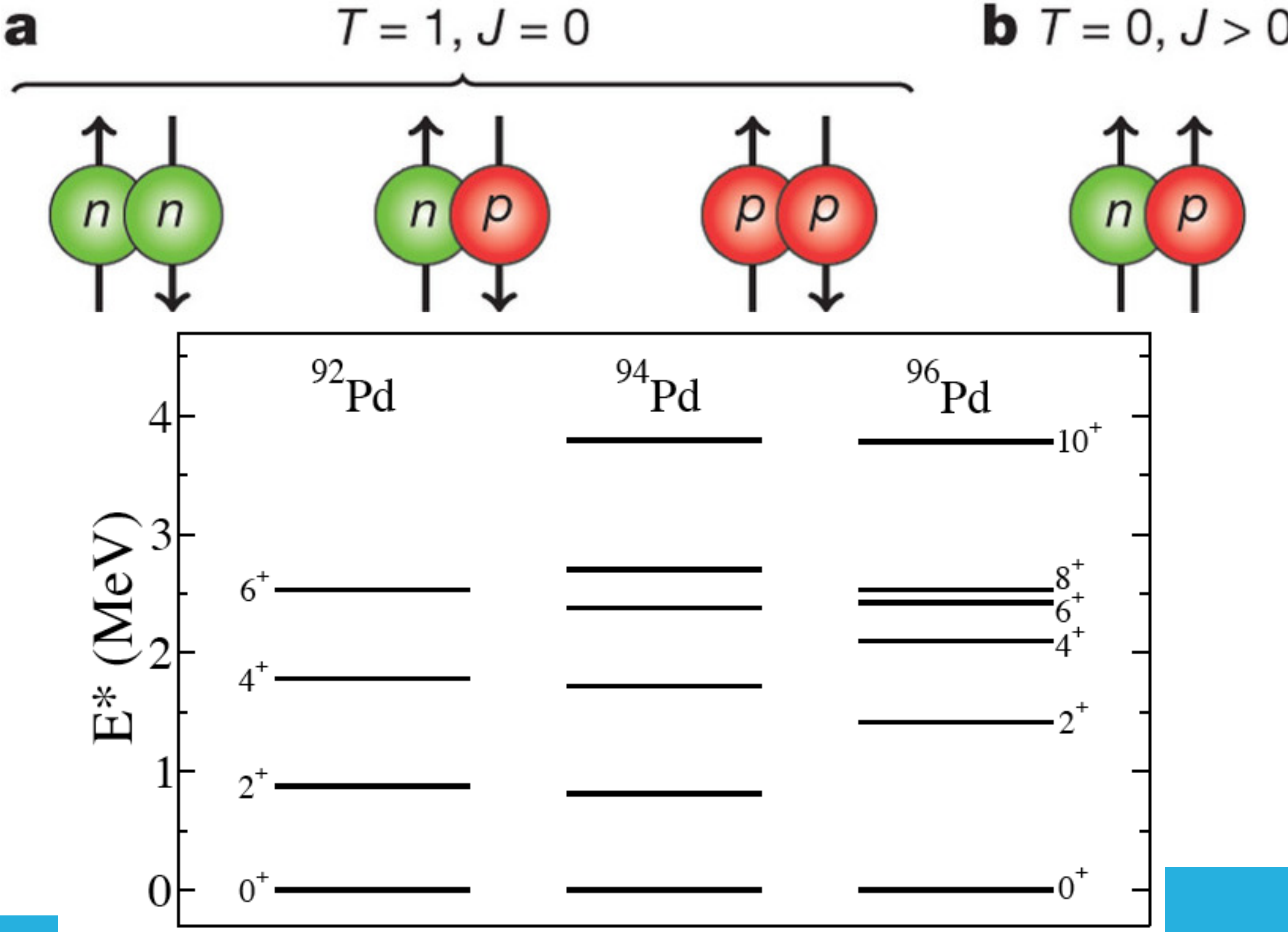
$$\langle 12|j_p^2;JM\rangle_a = N\big(1+(-1)^J\big)\langle 12|j_p^2;JM\rangle \tag{4.10}$$

J must be even and $N = 1/2$ (since $\langle j_p^2;JM|j_p^2;JM\rangle = 1$). Otherwise $N = 1/\sqrt{2}$.

As a consequence of the generalized Pauli principle a symmetric space-spin wave function must be combined with an antisymmetric isospin function or *vice versa.* In both cases the complete wave function is antisymmetric under the interchange of all coordinates of the two particles. This leads to the general statement that one obtains allowed two-particle states $|(lj)^2\rangle_{JT}$ only for

$$J + T = \text{odd}$$

$$1-(-1)^{J+T}$$



Nature 469, 68–71 (06 January 2011)