Rotational model and nuclear deformation

May 2, 2016

The general Hamiltonian corresponding to the motion of the A=N+Z nucleons in a nucleus is

$$H = \sum_{i=1}^{A} \frac{p_i^2}{2m_i} + \sum_{i<j=1}^{A} V_{ij}$$

$$H = \sum_{i=1}^{A} (\frac{p_i^2}{2m_i} + U(r_i)) + \sum_{i<j=1}^{A} V_{ij} - \sum_{i=1}^{A} U(r_i)$$

Atomic nucleus is a many-body system with great complexity. Although Quantum mechanics still governs its behavior, the forces are complicated and cannot, in fact, be written down explicitly in full detail. One has to rely on the construction of nuclear models

Different model views

Independent particle model

In the previous sessions we have considered the nucleus as a conglomerate of neutrons and protons moving freely in a central potential but satisfying the Pauli principle. It is the basis of any microscopic nuclear models.

Collective model

In the other extreme we have the collective model, where the individual nucleons form a compact entity. The Collective Model emphasizes the coherent behavior of all of the nucleons. Among the kinds of collective motion that can occur in nuclei are rotations or vibrations that involve the entire nucleus. A common feature of systems that have rotational spectra is the existence of a “deformation”, by which is implied a feature of anisotropy that makes it possible to specify an orientation of the system as a whole.

Deformed single-particle model

Variety of nuclear collective motions

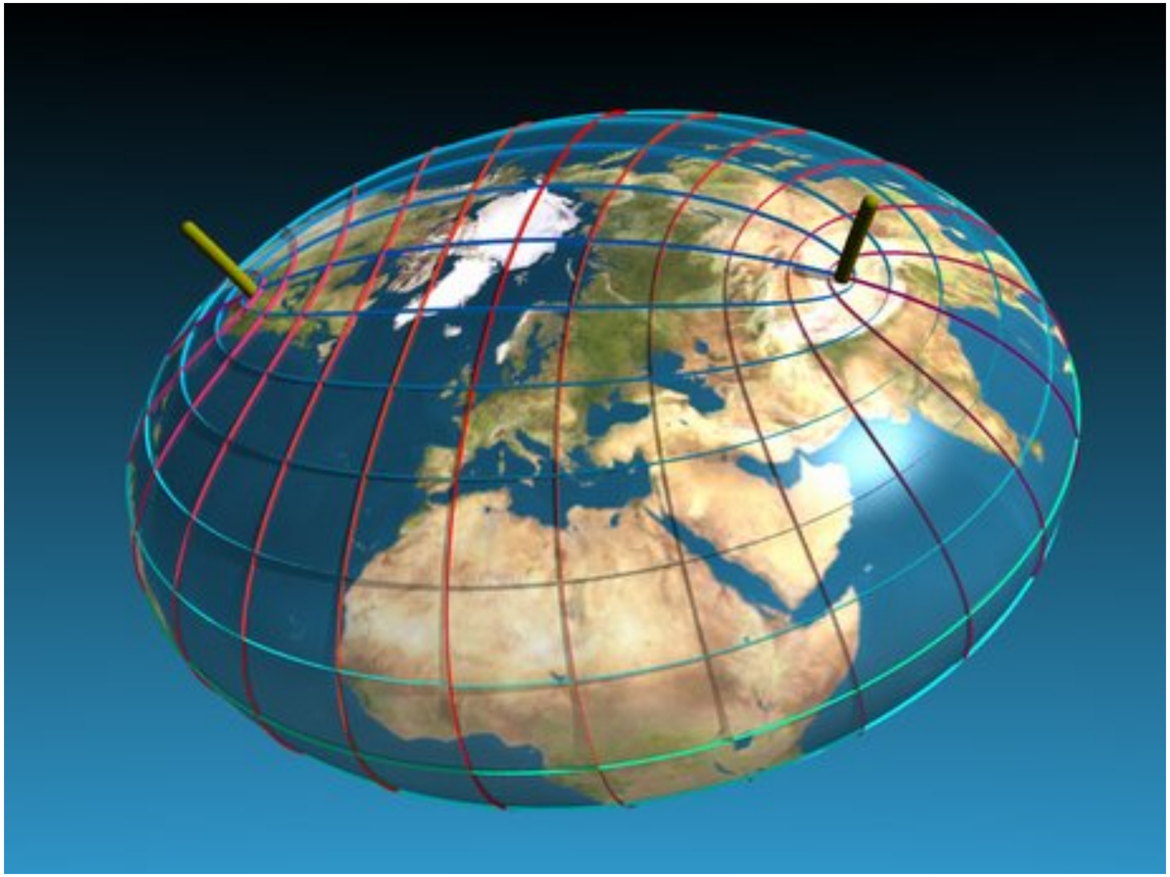
The single-particle shell model can not properly describe the excited states of nuclei. The excitation spectra of nuclei show characteristic of collective motions,

- ***Rotations;***
- Surface vibrations (quadrupole, octupole, hexadecupole, …);
- Fission (large-amplitude collective motion);
- Giant resonances (proton-neutron displacements,monopole, dipole, quadrupole, …)
- Scissors mode (proton-neutron angular displacement)
- Pygmy resonance (n-rich nuclei, vibration of neutron halo / skin with respect to the core)

In this course we will concentrate on simple descriptions of nuclear rotation.

Looking for all the world like little kids at recess, two of the twentieth century's greatest physicists (both won the Nobel Prize) watch a spinning tippy-top in fascination during a break at the 1954 inauguration of the Institute of Physics, Lund, Sweden. Wolfgang Pauli (1900–1958), on the left, was a deep mathematical theoretician, while Niels Bohr (1885–1962) was more of an intuitionist, yet the physics of the everyday schoolyard top straddled the purely mathematical and the experimental to embrace the imaginations of both men. Photograph courtesy of the AIP Emilio Segrè Visual Archives, the Margrethe Bohr Collection.

Planet Earth is triaxial

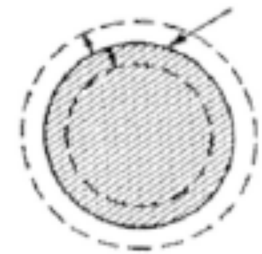


The Earth's equator is an ***ellipse*** rather than a circle

Types of Multipole Deformatiions

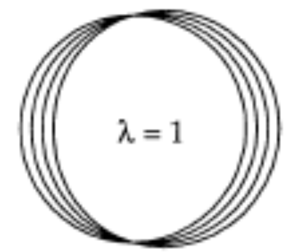
The monopole mode

$$Y_{00} = \frac{1}{4\pi} \longrightarrow R = R(\theta,\varphi,t) = R_0(1 + \sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda} \alpha^*_{\lambda\mu}(t) Y_{\lambda\mu}(\theta,\varphi))$$



The associated excitation is the so-called breathing mode of the nucleus. A large amount of energy is needed for the compression of nuclear matter and this mode is far too high in energy.

The dipole mode



Dipole deformations, to lowest order, do not correspond to a deformation of the nucleus but rather to a shift of the center of mass, i.e. a translation of the nucleus, and should be ***disregarded*** for nuclear excitations since translational shifts are spurious.

$$\vec{R}_{cm} = \frac{\int \vec{r}\rho(\vec{r})d^3r}{\int \rho(\vec{r})d^3r}$$

The center of mass of nucleus with "dipole deformation"

$$\vec{R}_{cm} = \frac{\int \vec{r}\rho(\vec{r})d^3r}{\int \rho(\vec{r})d^3r}$$

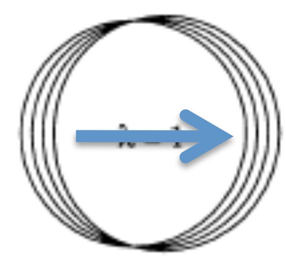
For the sphere

$$R_{cm} = 0$$

$$\vec{r} = \vec{r}_0 + \vec{r}_\mu$$

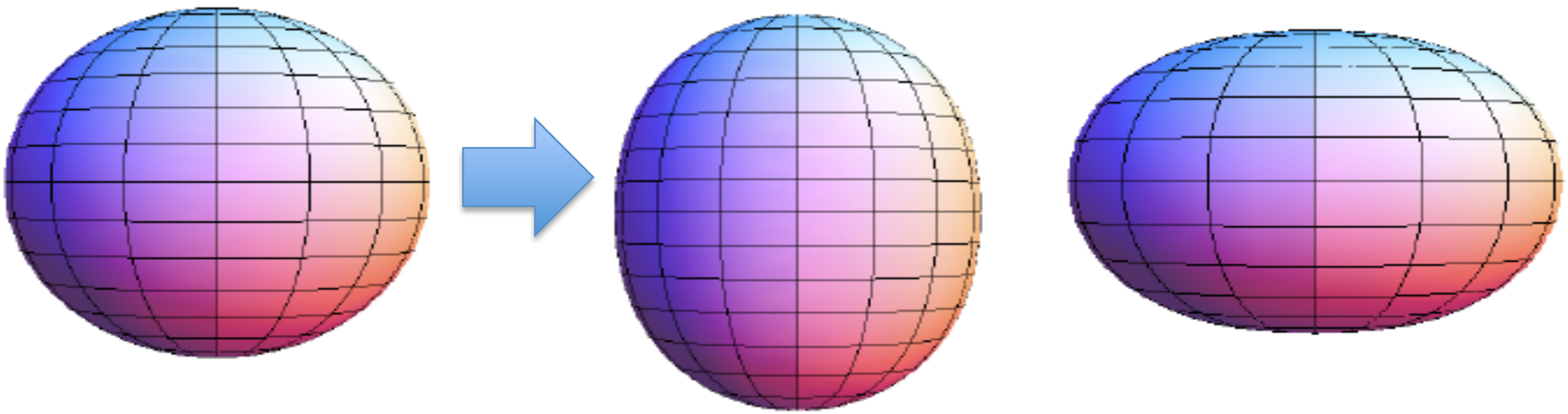
$$\vec{r}_\mu = \alpha_{1\mu} r_0 Y_{1\mu}(\theta,\varphi)$$

$$R_{cm,\mu} = \alpha_{1\mu} R_0$$



The quadrupole mode $\lambda = 2$

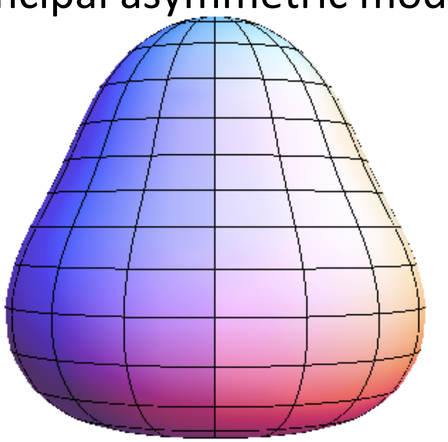
The most important nuclear shapes and collective low energy excitations of atomic nuclei.

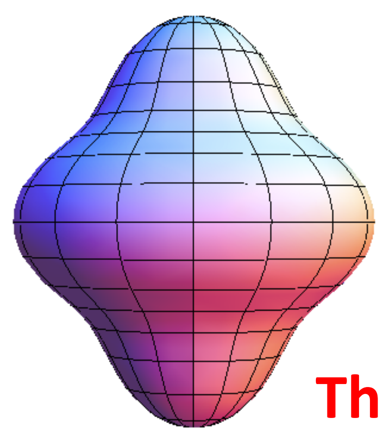


Spherical Prolate Obolat
e

The octupole mode $\lambda = 3$

The principal asymmetric modes of the nucleus associated with negative-parity bands.

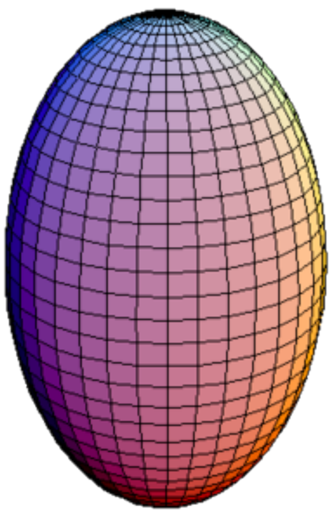


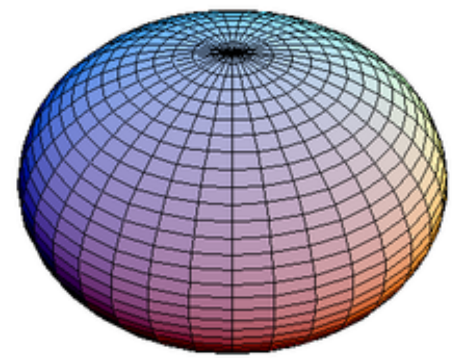


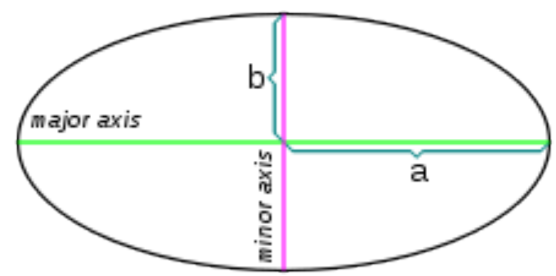
The hexadecupole mode $\lambda = 4$

A prolate spheroid (American football) is a spheroid in which the polar axis is greater than the equatorial diameter. The volume of a prolate spheroid is $V = \frac{4}{3}\pi a^2 b$ where *b* is the polar radius, and *a* is the equatorial radius.

An oblate spheroid (pancake) is a rotationally symmetric ellipsoid having a polar axis shorter than the diameter of the equatorial radius.







http://en.wikipedia.org/wiki/Prolate_spheroid
http://en.wikipedia.org/wiki/Oblate_spheroid

How are they related to the spherical amplitudes $\alpha_{2\mu}$

Hill-Wheeler coordinates

$$a_0 = \beta\cos\gamma \qquad a_2 = \frac{1}{\sqrt{2}}\beta\sin\gamma$$

$$\sum_\mu |\alpha_{2\mu}|^2 = \sum_\mu |\alpha'_{2\mu}|^2 = a_0^2 + 2a_2^2 = \beta^2$$

❖ **Consider the nuclear shapes in the principal axis system (x',y',z'), i.e. calculate the cartesian components as a function of γ for fixed β:**

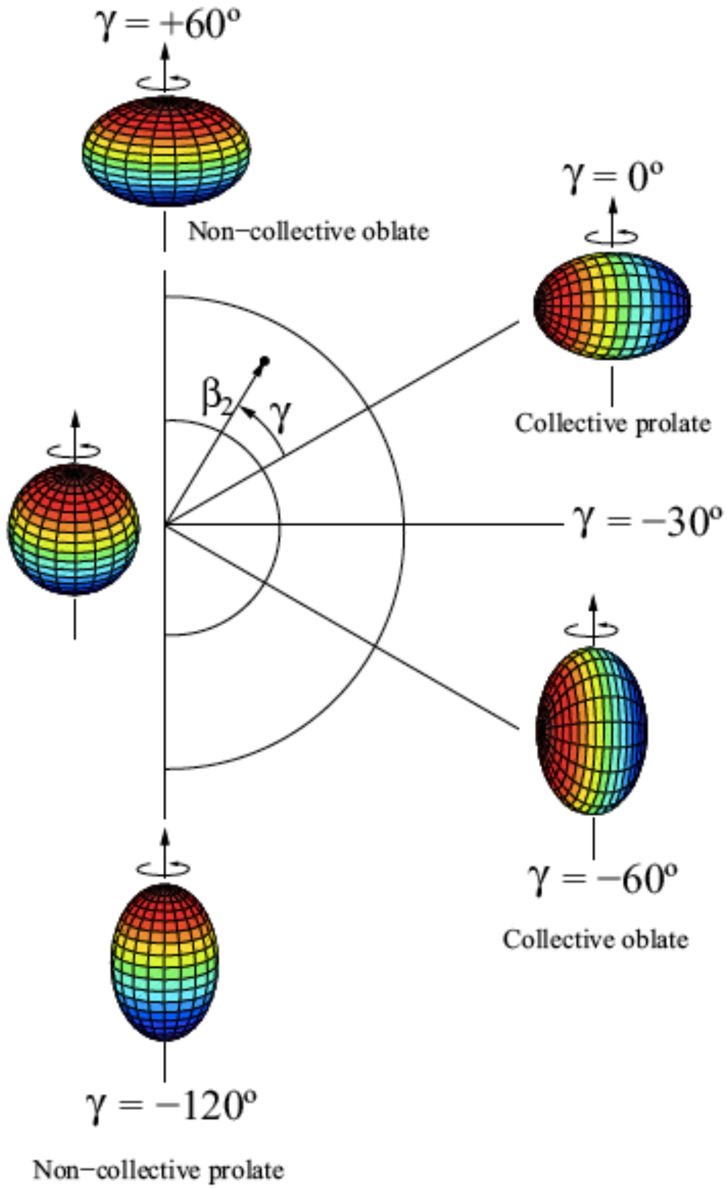
$$\alpha'_{z'z'} = \frac{\sqrt{6}}{3}\sqrt{\frac{15}{8\pi}}a_0 = \sqrt{\frac{5}{4\pi}}\beta\cos\gamma$$

$$\alpha'_{x'x'} = \sqrt{\frac{15}{8\pi}}(a_2 - \frac{1}{\sqrt{6}}a_0) = \sqrt{\frac{5}{4\pi}}\beta\cos(\gamma - \frac{2\pi}{3})$$

$$\alpha'_{y'y'} = \sqrt{\frac{5}{4\pi}}\beta\cos(\gamma - \frac{4\pi}{3})$$

or

$$\delta R_k = \sqrt{\frac{5}{4\pi}}\beta\cos(\gamma - \frac{2k\pi}{3}) \qquad k = 1,2,3 \quad for \quad x',y',z'$$



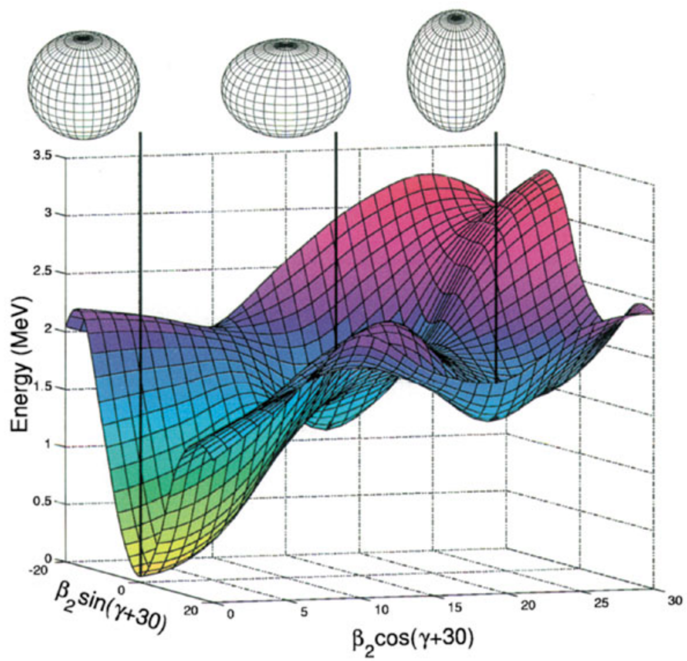
✧The nucleus is said to be *prolate when* two of the principal axes (x,y) are of the same length while the third axis (z) is longer.

✧If the third axis is shorter than the two equal principal axes, the nucleus is said to have an *oblate shape.*

✧γ=0◦ and γ=60◦ correspond to prolate and oblate shapes respectively. Completely triaxial shapes have γ=30◦

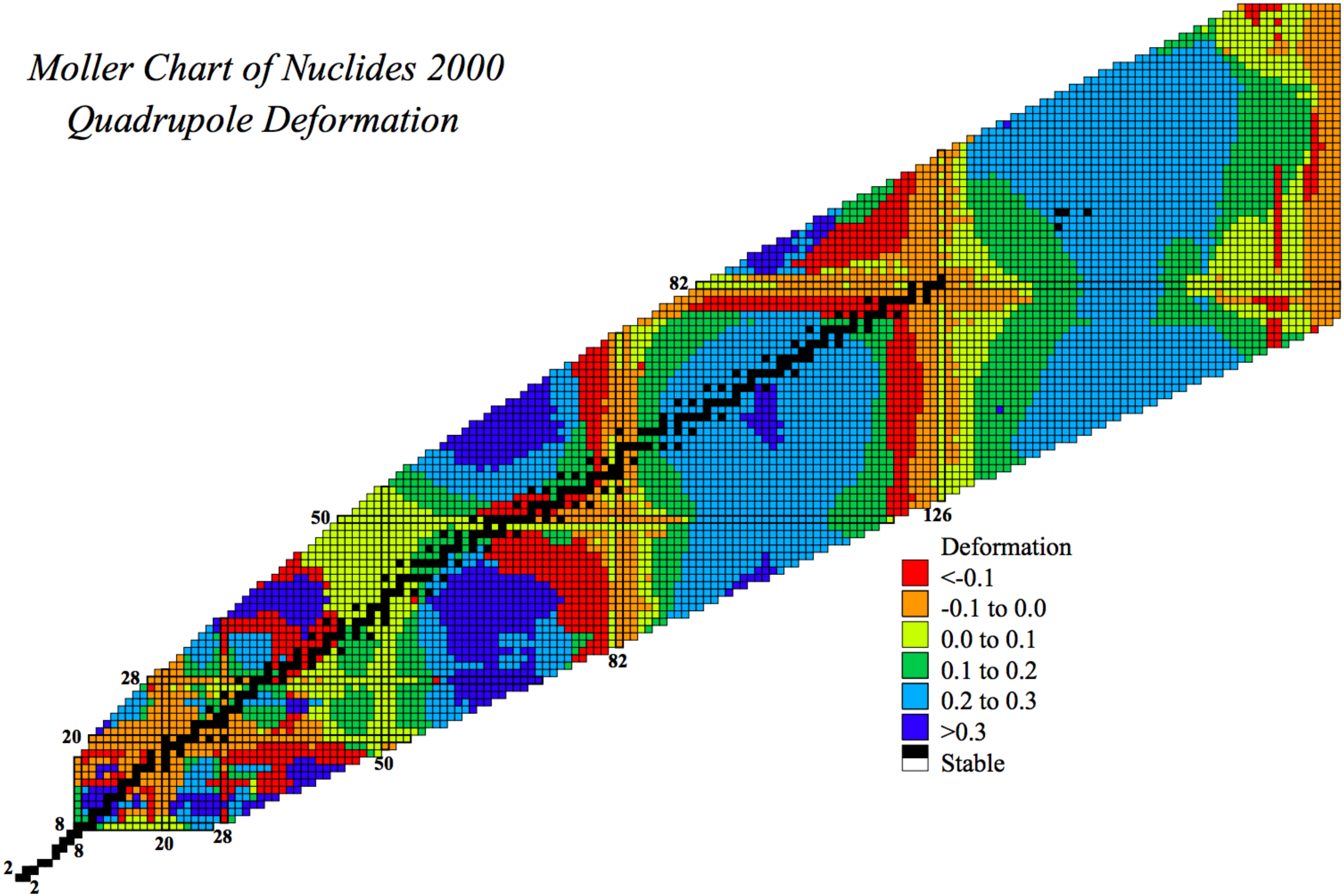
Description of the quadrupole deformation

Thus, the quadrupole deformation may be described either in a laboratory-fixed reference frame through the spherical tensor $a_{2\mu}$, or, alternatively, by giving the deformation of the nucleus with respect to the principal axis frame using the parameters **(a_0 ,a_2) or (beta,gamma)** and the Euler angles indicating the instantaneous orientation of the body-fixed frame.



http://ie.lbl.gov/systematics/chart_thb2.pdf

Moller Chart of Nuclides 2000
Quadrupole Deformation



As known from classical mechanics, the degrees of freedom of a rigid rotor are the **three Euler angles**, which describe the orientation of the body-fixed axes in space. **A classical rotor can rotate about any of its axis.**

The energy of a classical rotor can be described by

$$E = \frac{1}{2} J \omega^2$$

where J is the moment of inertia. Classically the angular momentum is given by.

$$l = J\omega$$

For the expression for the energy

$$E = \frac{1}{2} \frac{l^2}{J}$$

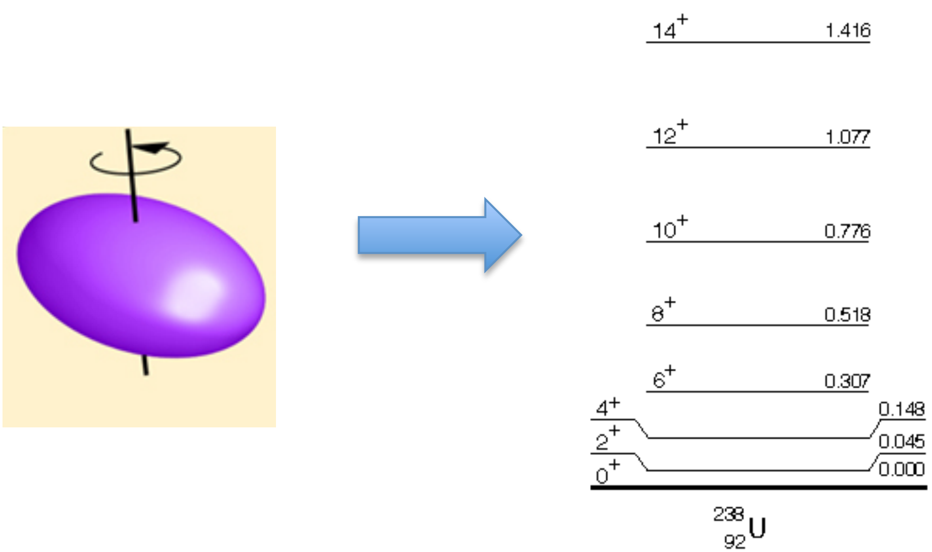
Collective rotation

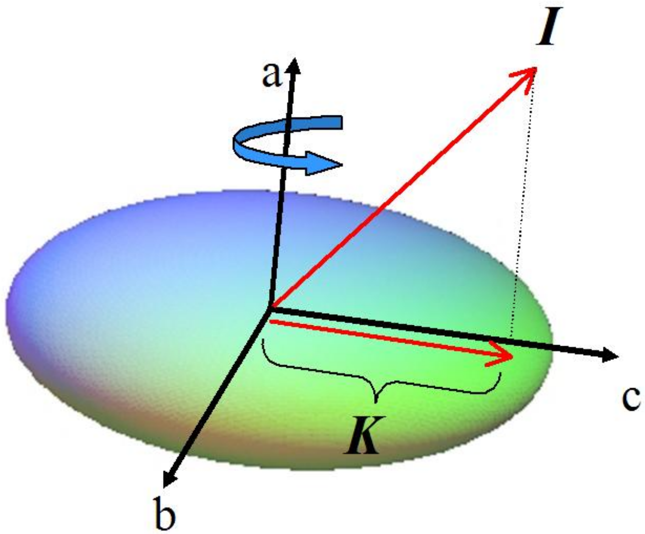
Rotation is a collective mode of excitation of a deformed nucleus found in different regions of the nuclear chart. This feature allows for the possibility to excite the nucleus by gaining rotational energy around an axis defined to be perpendicular to the symmetry axis.

A spherical nucleus has no rotational excitations at all !

In quantum mechanics the case is different. If the nucleus has rotational symmetries and no internal structure. For example, a spherical nucleus cannot rotate, because any rotation leaves the surface invariant and thus by definition does not change the quantum-mechanical state (and energy). This in turn implies that only a deformed nucleus can be said to be rotating.

A nucleus with axial symmetry cannot rotate around the axis of symmetry!



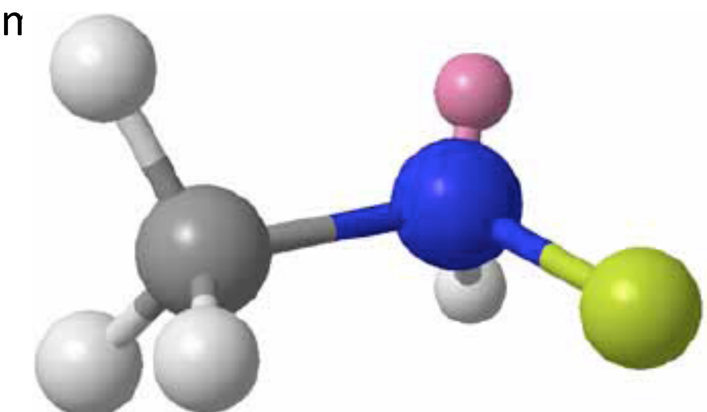


A prolate deformed nucleus

In fact

Nuclei are not always spherical!

In a molecule, as in a solid body, the deformation reflects the highly anisotropic mass distribution, as viewed from the intrinsic coordinate frame defined by the equilibrium positions of the nuclei. In the nucleus, the rotational degrees of freedom are associated with the deformations in the nuclear equilibrium structure.



molecule

In the quantum mechanical limit the squared angular momentum observable has the form

$$\boldsymbol{J}^2 = \hbar^2 I(I+1)$$

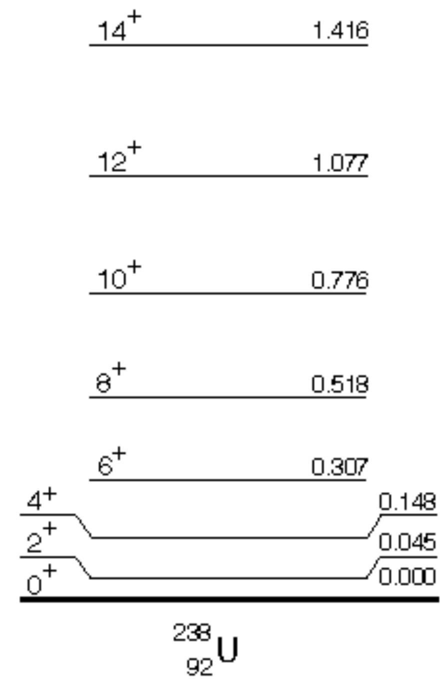
The Hamiltonian is

$$H = \frac{\boldsymbol{J}^2}{2\mathfrak{J}}$$

It gives the following formula for describing the energy levels of a rigid deformed rotor

$$E = \frac{\hbar^2}{2\mathcal{J}} I(I+1)$$

where I is the spin is of the state and **J is the static moment of inertia**.



Spin/parity I^π	0^+	2^+	4^+	6^+	8^+
Energy E	0	$6\frac{\hbar^2}{2J}$	$20\frac{\hbar^2}{2J}$	$42\frac{\hbar^2}{2J}$	$72\frac{\hbar^2}{2J}$
E_{I^π}/E_{2^+}	0	1	3.33	7	12

Quantum quadrupole axial rotor: ^{178}Hf

- Let us look into the lowest energy excitations in ^{178}Hf

Spin/parity I^π	0^+	2^+	4^+	6^+	8^+
Energy E [keV]	0	93.2	306.6	632.2	1058.6
E_{I^π}/E_{2^+}	0.00	1.00	3.29	6.78	11.36

- If we compare with the prediction of the rotor model we see a pretty good agreement (and small deviations to be discussed later).

Spin/parity I^π	0^+	2^+	4^+	6^+	8^+
Energy E	0	$6\frac{\hbar^2}{2J}$	$20\frac{\hbar^2}{2J}$	$42\frac{\hbar^2}{2J}$	$72\frac{\hbar^2}{2J}$
E_{I^π}/E_{2^+}	0.00	1.00	3.33	7.00	12.00

The Nobel Prize in Physics 1975

Aage N. Bohr, Ben R. Mottelson, James Rainwater

The Nobel Prize in Physics 1975

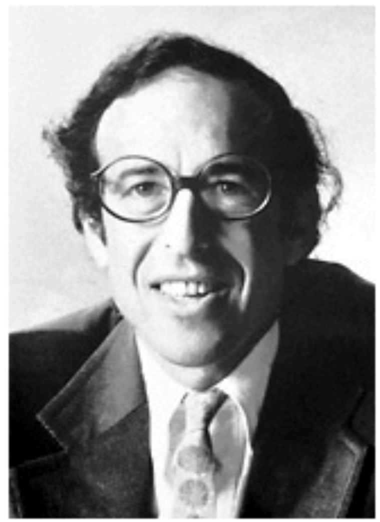
Nobel Prize Award Ceremony

Aage N. Bohr

Ben R. Mottelson

James Rainwater

Aage Niels Bohr



Ben Roy Mottelson

Leo James Rainwater

The Nobel Prize in Physics 1975 was awarded jointly to Aage Niels Bohr, Ben Roy Mottelson and Leo James Rainwater *"for the discovery of the connection between collective motion and particle motion in atomic nuclei and the development of the theory of the structure of the atomic nucleus based on this connection"*.

http://www.nobelprize.org/nobel_prizes/physics/laureates/1975/

From classical mechanics it is known that three angles are needed to define the position of a rigid body with fixed center of mass. These are called Euler angles. The energy of the rotating rigid body, with the center of mass fixed at the center of coordinates, is

$$E = \frac{J_{x'}^2}{2\mathfrak{I}_{x'}} + \frac{J_{y'}^2}{2\mathfrak{I}_{y'}} + \frac{J_{z'}^2}{2\mathfrak{I}_{z'}}$$

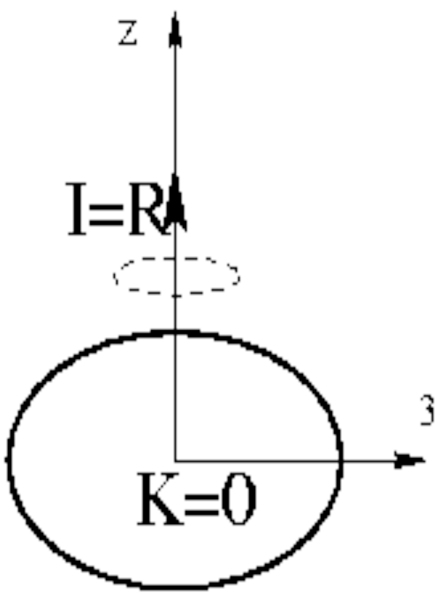
where $\mathfrak{I}_{x'}$ is the x' component of the moment of inertia and $J_{x'}$ is the corresponding angular momentum component.

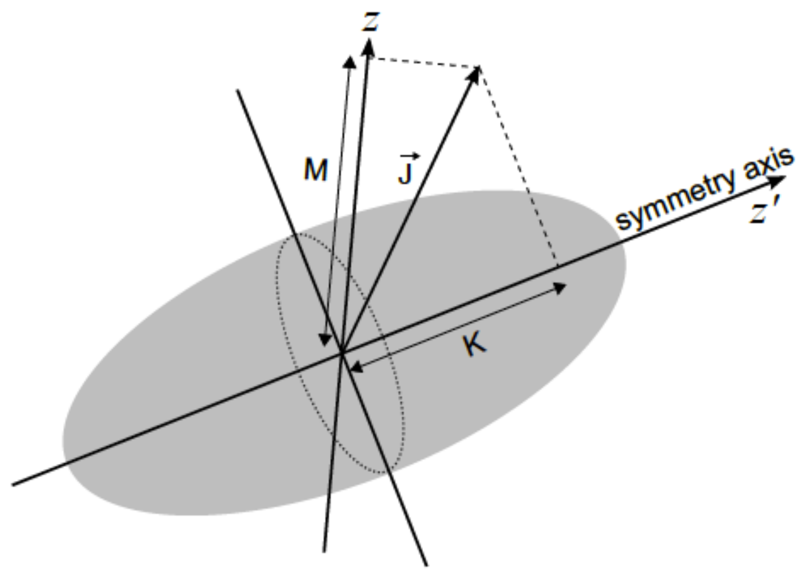
We will assume that the rigid body has cylindrical symmetry along the z' axis. Therefore the component $J_{z'}$ of the angular momentum, which is usually denoted by the letter K, is conserved. This symmetry also implies that $\mathfrak{I}_{x'} = \mathfrak{I}_{y'}$. We will use the symbol $\mathfrak{I}$ to denote this moment of inertia.

Angular momentum quantum numbers describing rotational motion in three dimensions.

The *z axis belongs to a coordinate system fixed in the laboratory, while the 3* axis is part of a body-fixed coordinate system

Quantum mechanically the component J_3 = K is conserved. One has the total angular momentum is J = I, since it is a constant of the motion. Jz = M is a constant of the motion. If the system possesses axial symmetry, The projection *on the symmetry axis is also a constant of the motion,* J_3 = K.





The Hamiltonian

$$H = \frac{\boldsymbol{J}^2}{2\mathfrak{I}}$$

In quantum mechanics there is no rotation along the symmetry axis, therefore

$$H = \frac{J_{x'}^2 + J_{y'}^2}{2\mathfrak{I}} = \frac{J^2 - J_{z'}^2}{2\mathfrak{I}}$$

The eigenvalues corresponding to this Hamiltonian are

$$E(J,K) = \hbar^2 \frac{J(J+1) - K^2}{2\mathfrak{I}}$$

$$D^J_{MK}(\theta,\phi,\varphi) = \langle \theta\phi\varphi | JMK \rangle$$

which are called "d-functions". They satisfy the eigenvalue equations,

$$\begin{aligned}
\boldsymbol{J}^2 D^J_{MK}(\theta,\phi,\varphi) &= \hbar^2 J(J+1) D^J_{MK}(\theta,\phi,\varphi), \\
J_z D^J_{MK}(\theta,\phi,\varphi) &= \hbar M D^J_{MK}(\theta,\phi,\varphi), \\
J_{z'} D^J_{MK}(\theta,\phi,\varphi) &= \hbar K D^J_{MK}(\theta,\phi,\varphi)
\end{aligned}$$

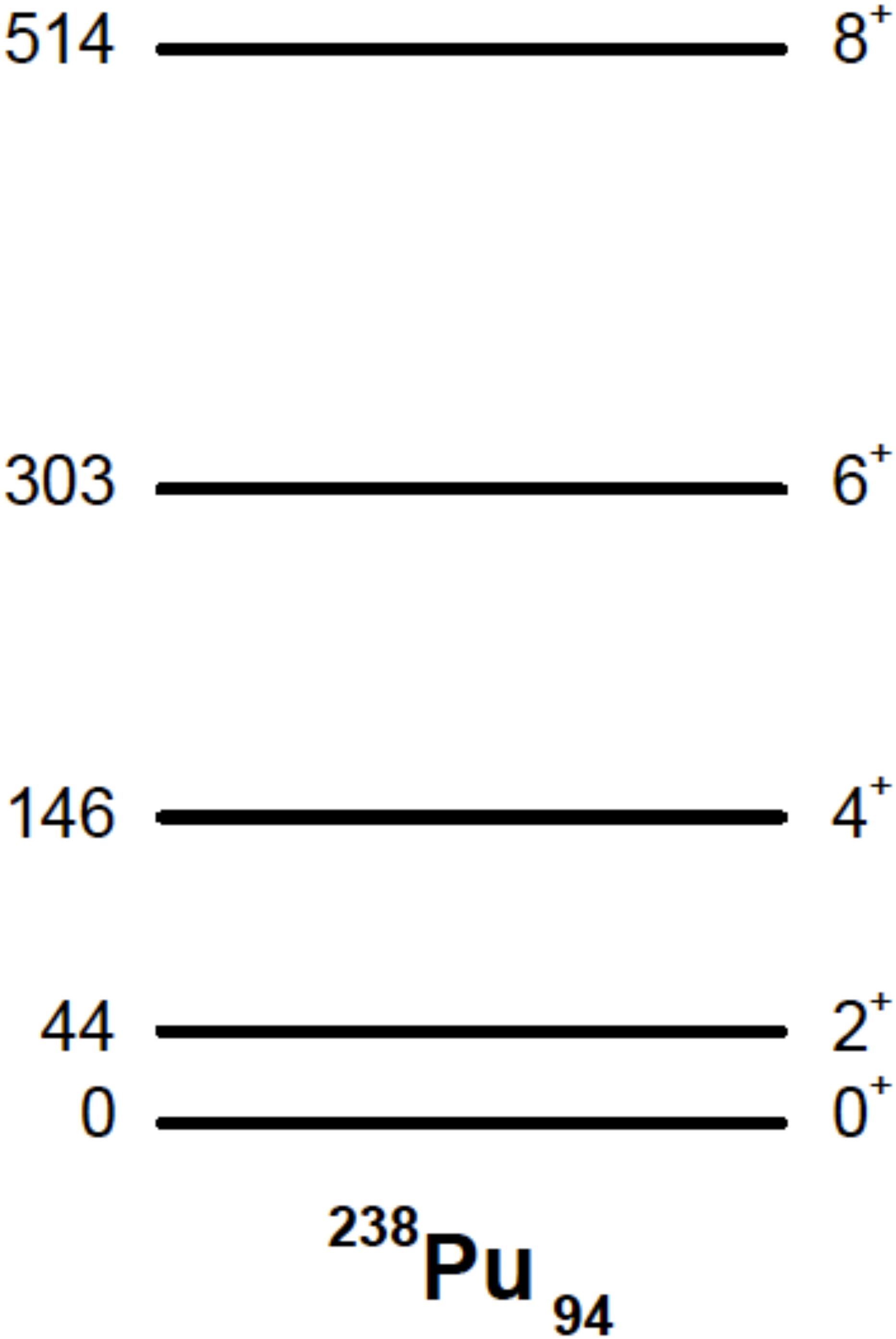
By assuming cylindrical symmetry, we have

$$\langle \theta\phi\varphi | JMK \rangle = c \left(D^J_{MK} + (-1)^J D^J_{M-K} \right)$$

where c is a constant

The lowest lying of these bands is the one corresponding to K = 0,

$$E(J,0) = \hbar^2 \frac{J(J+1)}{2\mathfrak{I}}$$



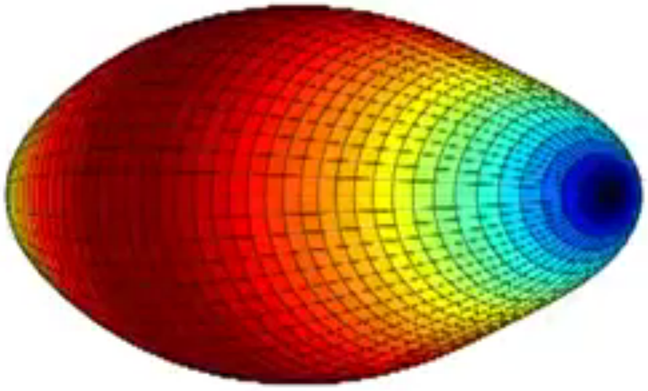
$$E(J,0) = \hbar^2 \frac{J(J+1)}{2\Im}$$

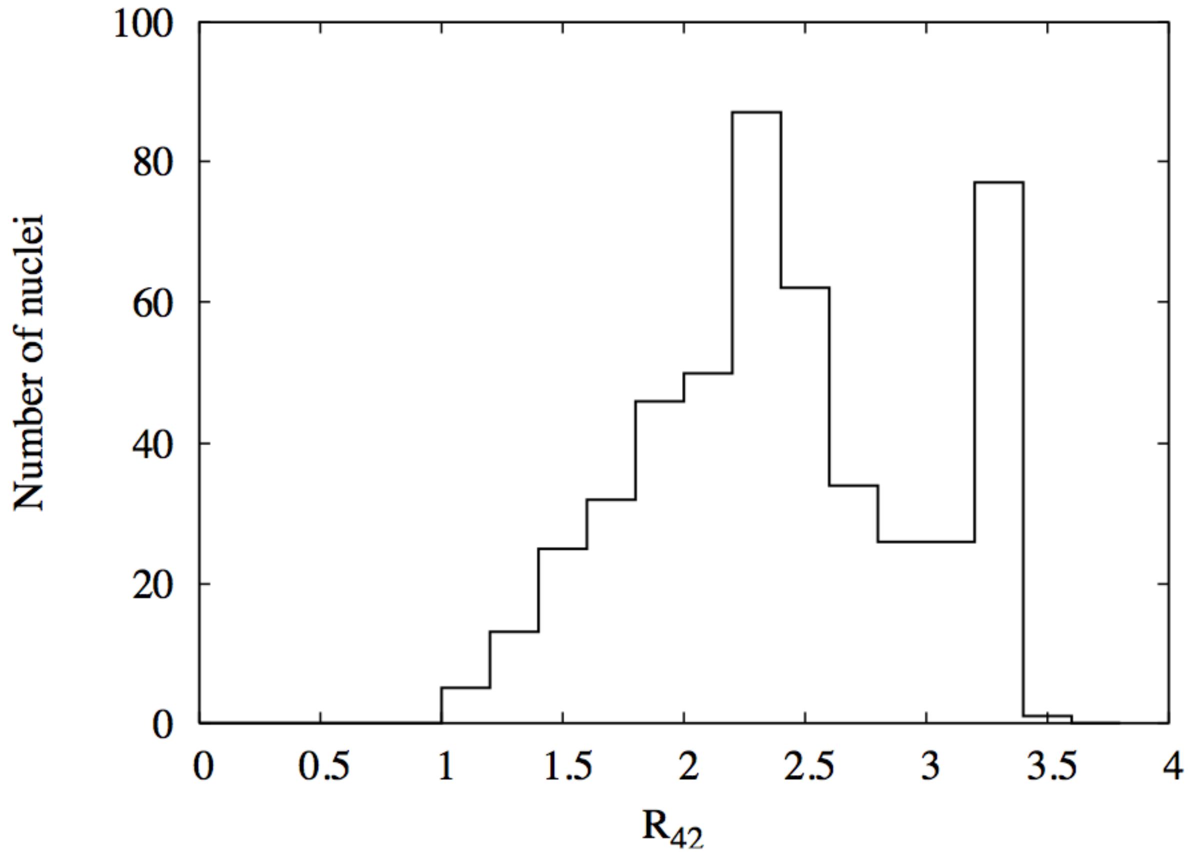
$$\Im = \hbar^2 \frac{J(J+1)}{2E(2,0)}.$$

$$E(J,0) \;=\; E(2,0)\;\; J(J+1)/6$$

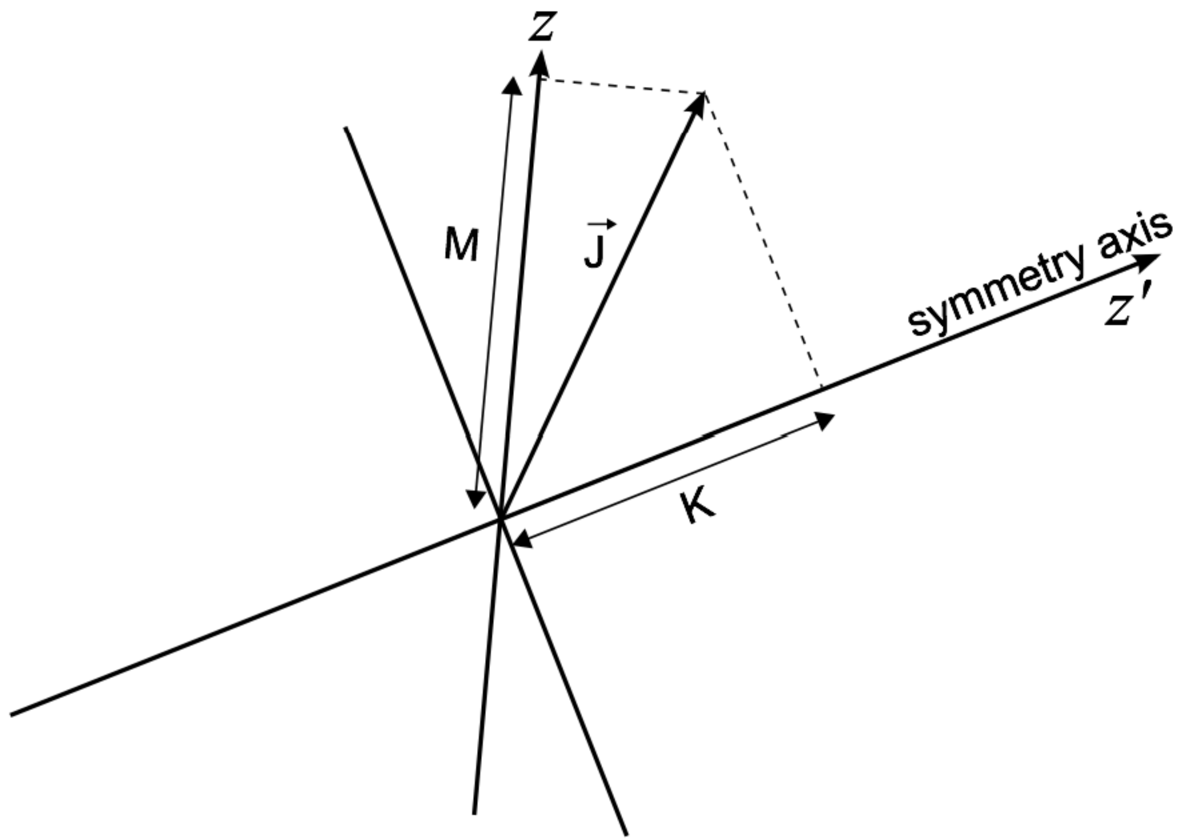
$E(4,0)$=147 keV, $E(6,0)$=308

E(8,0)= 528 keV





100
80
60
40
20
0
0
0.5
1
1.5
2
2.5
3
3.5
4
Number of nuclei
R42



z
M
$\vec{J}$
symmetry axis
z′
K

The deformed single-particle model

In above description, we consider the rotational motion of the system as a whole and neglected the internal motion with respect to the body-fixed coordinate frame.

The starting point for the description of the intrinsic degrees of freedom in deformed nuclei is the analysis of one-particle motion in non-spherical potentials.

In the following we discuss the deformed single-particle potential and the associated one-particle quantum states

The generalization of the phenomenological shell model to deformed nuclear shapes was first given by S. G. Nilsson in 1955, so this version is often referred to the Nilsson model.

Reading

The Nilsson Model and Sven Gösta Nilsson
Ben Mottelson, Phys. Scr. T125 (2006)
http://iopscience.iop.org/1402-4896/2006/T125/E02/pdf/physscr6_t125_e02.pdf

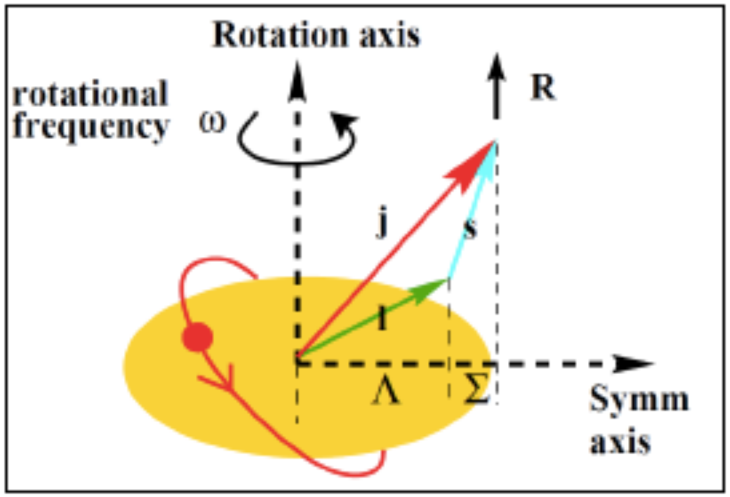
'Another impressive indication of the validity of the independent particle model is the immense success of the Nilsson scheme. We all know the famous level scheme and the popularity of his paper—I am sure this is the one paper which one finds on the desk of every nuclear physicist.'

The numerical diagonalization of the matrices involved (up to dimensions 7 × 7) required that Sven Gösta travel to Stockholm in order to exploit the power of the BESK computer (at that time the largest available for scientific computation in Sweden).

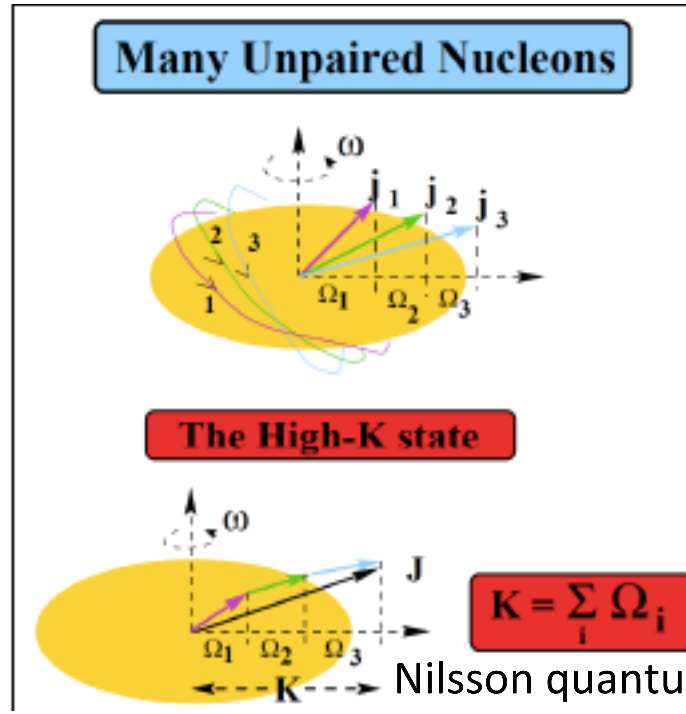
Dimension we can handle today
$10^{10} \times 10^{10}$

http://www.pdc.kth.se/resources/computers/lindgren

The physics behind



$$\Omega = \Sigma + \Lambda$$



Nilsson quantum numbers

The projection, K is the intrinsic single particle spin of the band-head state.

Reminder: 3D isotropic harmonic oscillator

One-dimensional harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,, \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x}\,.$$

3D isotropic harmonic oscillator

$$V(r) = \frac{1}{2}\mu\omega^2 r^2,$$

The Hamiltonian can be written as

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 R^2 = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + \frac{1}{2}m\omega^2(X^2 + Y^2 + Z^2) = H_x + H_y + H_z\,.$$

$$H_i|\phi_{ni}\rangle = E_{ni}|\phi_{ni}\rangle = (n_i + \tfrac{1}{2})\hbar\omega|\phi_{ni}\rangle$$

$$H|\psi_{n_x,n_y,n_z}\rangle = (n_x + n_y + n_z + \tfrac{3}{2})\hbar\omega|\psi_{n_x,n_y,n_z}\rangle$$

$$\{|\psi_{n_x,n_y,n_z}\rangle = |\phi_{n_x}\rangle \otimes |\phi_{n_y}\rangle \otimes |\phi_{n_z}\rangle\}$$

$$E_n = (n + \tfrac{3}{2})\hbar\omega \qquad n = n_x + n_y + n_z$$

The anisotropic harmonic oscillator

The Harmonic Oscillator potential can be generalized so as to be applicable to the deformed case.

The principal idea is to make the oscillator constants different in the different spatial directions:

$$H_{def} = -\frac{\hbar^2}{2m}\Delta + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

The condition of incompressibility of nuclear matter requires that the volume of the ellipsoid should be the same as that of the sphere and this imposes a condition on the oscillator frequencies:

$$\omega_x \omega_y \omega_z = \omega_0^3 \qquad \omega_x = \omega \frac{R_0}{a_x}; \cdots$$

If we assume that the nuclear z-axis (3-axis) is different from the extension along the x- and *y-axes, we may write the single-particle Hamiltonian in the form*

$$H = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{M}{2}\left[\omega_\perp^2\left(x^2+y^2\right) + \omega_z^2 z^2\right]$$

The anisotropy corresponds to the difference introduced between $\omega_\perp$ and ω_z. It is convenient to introduce an elongation parameter ε (Nilsson, 1955):

$$\omega_z = \omega_0(\varepsilon)\left(1 - \frac{2}{3}\varepsilon\right)$$

$$\omega_\perp = \omega_0(\varepsilon)\left(1 + \frac{1}{3}\varepsilon\right)$$

where $\omega_0(\varepsilon)$ is weakly ε-dependent, enough to conserve the nuclear volume (see below). The distortion parameter ε is obtained as $\varepsilon = (\omega_\perp - \omega_z)/\omega_0$. It is defined so that $\varepsilon > 0$ and $\varepsilon < 0$ correspond to so-called prolate and oblate shapes, respectively.

$$\varepsilon \approx \frac{3}{2}\left(\frac{5}{4\pi}\right)^{1/2} \beta_2 \approx 0.95\beta_2$$

For the spheroidal potential, the motion separates into independent oscillations along the 3 axis and in the (12) plane

$$\frac{1}{2}m\omega_\perp^2(x^2+y^2)+\frac{1}{2}m\omega_z^2 z^2$$
$$=\frac{1}{2}m\omega_0^2 r^2 - m\omega_0^2\beta\, r^2 Y_{20}(\theta)$$

The energy is

$$\varepsilon(n_3 n_\perp)=(n_3+\tfrac{1}{2})\hbar\omega_3+(n_\perp+1)\hbar\omega_\perp$$

where $n_\perp = n_1 + n_2$ is the number of quanta in the oscillations perpendicular to the symmetry axis.

The Nilsson model

Deformed HO potential with ls and l^2 corrections

$$H = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{M}{2}\left[\omega_\perp^2\left(x^2+y^2\right)+\omega_z^2 z^2\right]-C\boldsymbol{\ell}\cdot\mathbf{s}$$

$$-D\left(\ell^2-\left\langle \ell^2\right\rangle_N\right)$$

As mentioned in last section, the l^2 term lifts the degeneracy within each major oscillator shell in such a manner as to favor the states with large *l* .

The term $<l^2>_N$ **is a constant for each oscillator** shell chosen so that the average energy difference between shells is not affected by the $\mathbf{l^2}$ **term.**

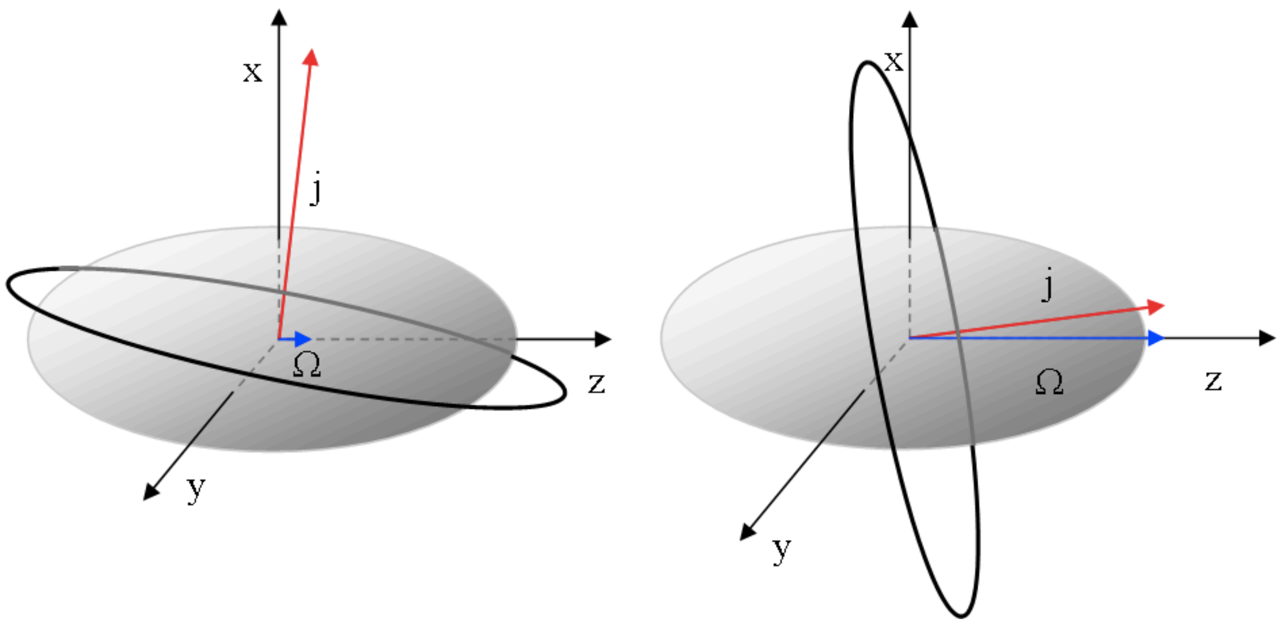
$$\langle \mathbf{l}^2\rangle_N = \tfrac{1}{2}N(N+3)$$

The axial symmetry of the nuclear potential imply that the parity *and the projection of the total angular momentum along* the symmetry axis, Ω ,are constants of the motion for the one-particle states.

One may classify the levels according to the cylindrical quantum numbers.

$$\Omega^{\pi}[Nn_z m]$$

where the projection of total angular momentum Ω, and the parity π are good quantum numbers while N, n_z and m are only approximate and may be determined for a given level only by looking at its behavior near the spherical state



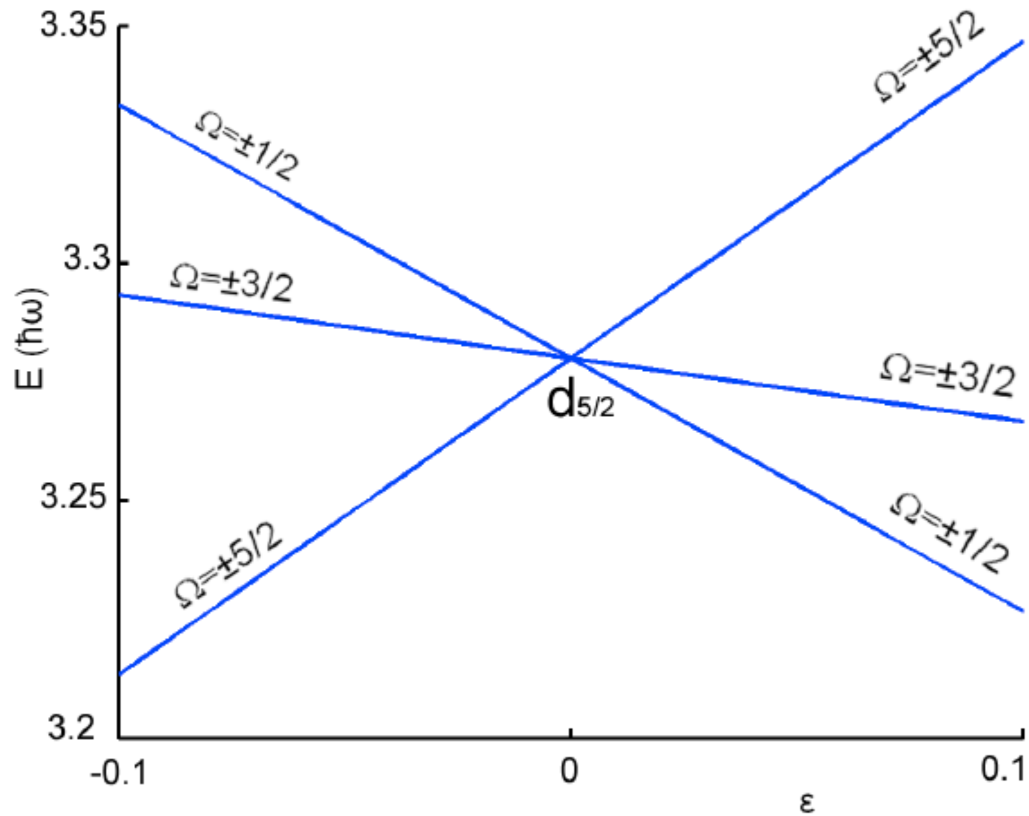
In the spherical case each j state is $(2j+1)$-fold degenerate. This degeneracy is removed by the perturbation h' to first order as

$$\langle N\ell sj\Omega|\varepsilon h'|N\ell sj\Omega\rangle = \frac{1}{6}\varepsilon M\omega_0^2\left\langle r^2\right\rangle \frac{3\Omega^2 - j(j+1)}{j(j+1)}$$

- Each spherical level labeled by $N(l_j)$ at $\varepsilon=0$, is split into $(2j+1)/2$ levels with

$$\Omega = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm j.$$

- **The remaining degeneracy means that each level can accommodate two nucleons.**
- **Orbits with lower Ω are shifted downwards for $\varepsilon>0$ (prolate) and upwards for $\varepsilon<0$ (oblate).**



Lowest part of the level diagram (Nilsson diagram) for the deformed shell model.

The single-particle energies are plotted as functions of deformation β_0 and are given in units of $\hbar\bar{\omega}_0$.

The quantum numbers Ω^{π} for the individual levels and l_j, for the spherical ones are indicated as well as the magic numbers for the spherical shape.

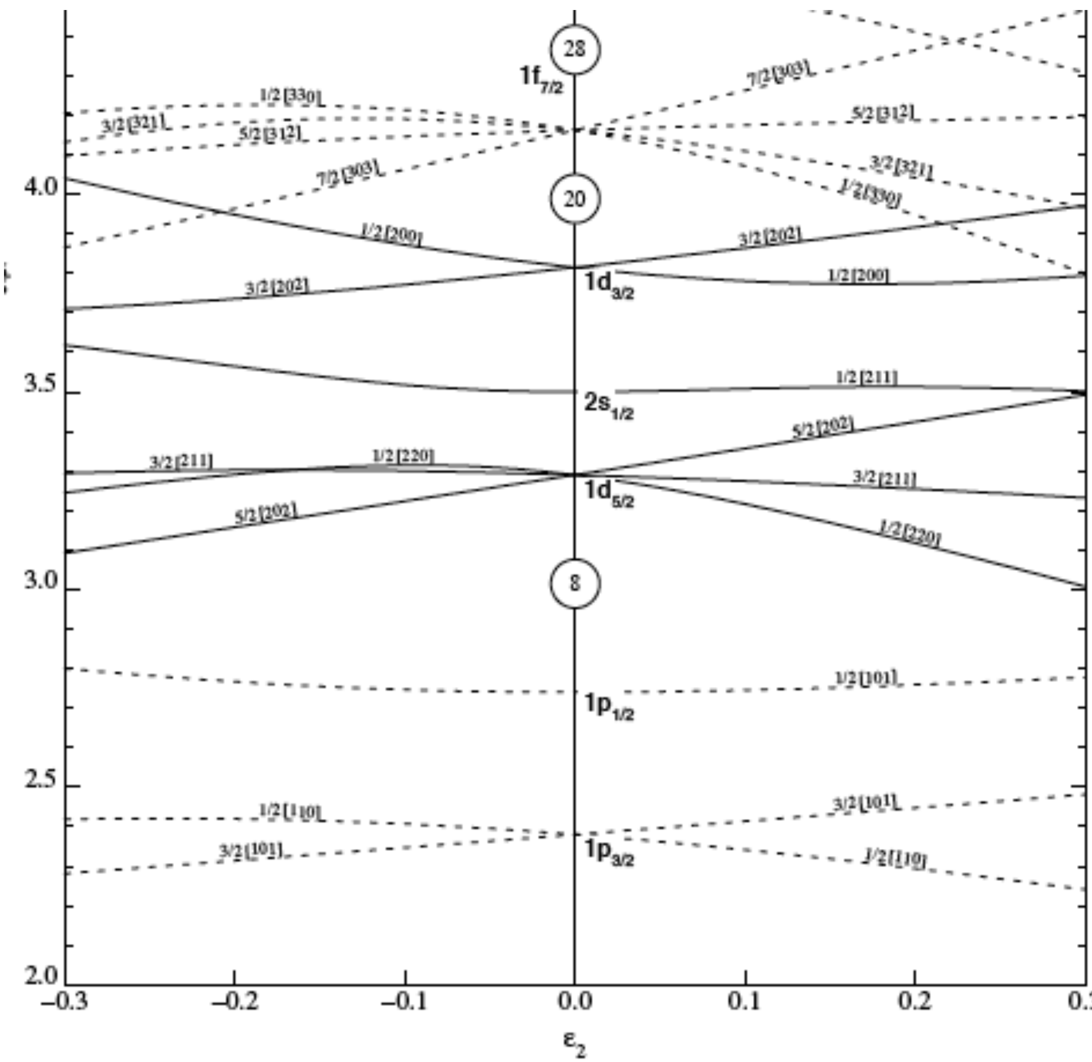


Figure 4. Nilsson diagram for protons or neutrons, Z or N ≤ 50 ($\varepsilon_4 = 0$).

$[Nn_3\Lambda\Omega]$

As for the three dimensional potential well the Nilsson model predicts that shells and shell gaps are modied by the deformation.

The main achievement of the Nilsson model is correct explanation of ground state spins and parities of a large number of nuclei, as well its ability to be expanded into a model for rotation of deformed odd-mass nuclei

Spin and Magnetic Moment of ^{33}Mg
3/2[321]
PRL 99, 212501 (2007)
31Mg: ½[200]
PRL 94, 022501 (2005)

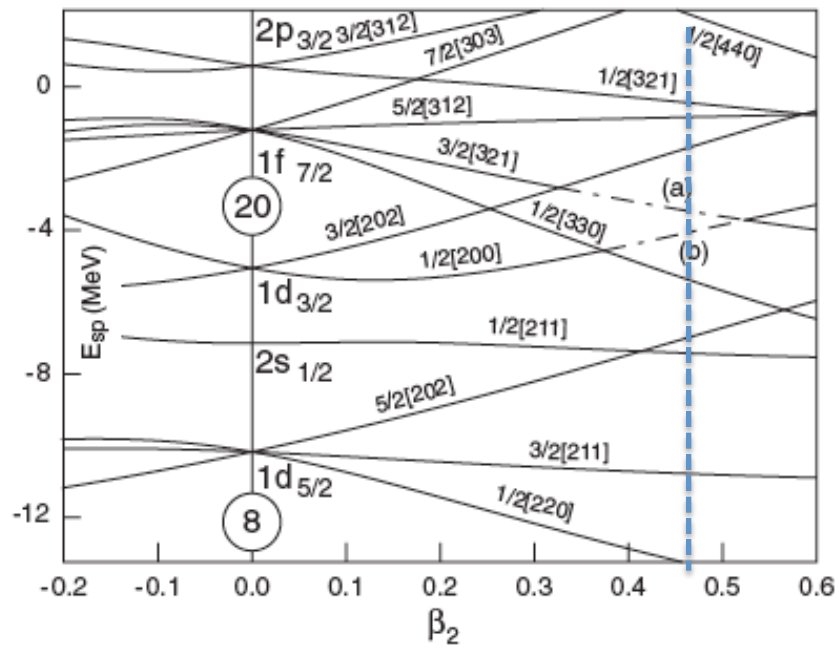


FIG. 3. Nilsson diagram in the $\nu(sd\text{-}1f_{7/2}\text{-}2p_{3/2})$ configuration space calculated using universal Woods-Saxon potential [26,27]. (a), (b) Odd-neutron occupation in the ground states of ^{33}Mg and ^{31}Mg, respectively.

http://www.sciencedaily.com/releases/2011/02/110202143800.htm