Chapter 12

Partial differential equations

12.1 Differential operators in $\mathbb{R}^n$

Differential operators

We recall the definition of the gradient of a scalar function as

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right)^T, \tag{12.1}$$

which we can interpret as the differential operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right)^T, \tag{12.2}$$

acting on the function $f = f(x)$, with $x \in \Omega \subset \mathbb{R}^n$. With this interpretation we express two second order differential operators, the *Laplacian* Δf,

$$\Delta f = \nabla^T \nabla f = \frac{\partial^2 f}{\partial x_1^2} + ... + \frac{\partial^2 f}{\partial x_n^2}, \tag{12.3}$$

and the *Hessian* Hf,

$$Hf = \nabla \nabla^T f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}. \tag{12.4}$$

For a vector valued function $f : \mathbb{R}^n \to \mathbb{R}^m$, we define the Jacobian matrix by

$$f' = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} (\nabla f_1)^T \\ \vdots \\ (\nabla f_m)^T \end{bmatrix}, \tag{12.5}$$

and for $m = n$, we define the *divergence* by

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + ... + \frac{\partial f_n}{\partial x_n}. \tag{12.6}$$

Partial integration in $\mathbb{R}^n$

For the scalar function $f : \mathbb{R}^n \to \mathbb{R}$, and the vector valued function $g : \mathbb{R}^n \to \mathbb{R}^n$, we have the following generalization of partial integration over $\Omega \subset \mathbb{R}^n$, referred to as *Green's theorem*,

$$(\nabla f, g) = -(f, \nabla \cdot g) + (f, g \cdot n)_\Gamma, \tag{12.7}$$

where we use the notation,

$$(v, w)_\Gamma = (v, w)_{L^2(\Gamma)} = \int_\Omega vw\, ds, \tag{12.8}$$

for the boundary integral, with $L^2(\Gamma)$ the Lebesgue space defined over the boundary Γ.

Sobolev spaces

The L^2 space for $\Omega \subset \mathbb{R}^n$, is defined by

$$L^2(\Omega) = \{v : \int_\Omega |v|^2\, dx < \infty\}, \tag{12.9}$$

where in the case of a vector valued function $v : \mathbb{R}^n \to \mathbb{R}^n$, we let

$$|v|^2 = \|v\|_2^2 = v_1^2 + ... + v_n^2. \tag{12.10}$$

To construct appropriate vector spaces for the variational formulation of partial differential equations, we need to extend L^2 spaces to include also derivatives. The *Sobolev space* $H^1(\Omega)$ is defined by,

$$H^1(\Omega) = \{v \in L^2(\Omega) : \frac{\partial v_i}{\partial x_j} \in L^2(\Omega), \forall i, j = 1, ..., n\}, \tag{12.11}$$

and we define

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v(x) = 0,\ x \in \Gamma\}, \tag{12.12}$$

to be the space with functions that are zero on the boundary Γ.

12.2 Poisson's equation

The Poisson equation

We now consider the *Poisson equation* for a function $u \in \mathcal{C}^2(\Omega)$,

$$-\Delta u = f, \quad x \in \Omega, \tag{12.13}$$

with $\Omega \subset \mathbb{R}^n$, and $f \in \mathcal{C}(\Omega)$. For the equation to have a unique solution we need to specify boundary conditions. We may prescribe *Dirichlet boundary conditions*,

$$u = g_D, \quad x \in \Gamma, \tag{12.14}$$

Neumann boundary conditions,

$$\nabla u \cdot n = g_N, \quad x \in \Gamma, \tag{12.15}$$

with $n = n(x)$ the outward unit normal on Γ_N, or a linear combination of the two, which we refer to as a *Robin boundary condition.*

Homogeneous Dirichlet boundary conditions

We now state the variational formulation of Poisson equation with homogeneous Dirichlet boundary conditions,

$$-\Delta u = f, \qquad x \in \Omega, \tag{12.16}$$
$$u = 0, \qquad x \in \Gamma, \tag{12.17}$$

which we obtain by multiplication by a test function $v \in V = H_0^1(\Omega)$ and integration over Ω, using Green's theorem, which gives,

$$(\nabla u, \nabla v) = (f, v), \tag{12.18}$$

since the boundary term vanishes as the test function is an element of the vector space $H_0^1(\Omega)$.

Homogeneous Neumann boundary conditions

We now state the variational formulation of Poisson equation with homogeneous Neumann boundary conditions,

$$-\Delta u = f, \qquad x \in \Omega, \tag{12.19}$$
$$\nabla u \cdot n = 0, \qquad x \in \Gamma, \tag{12.20}$$

which we obtain by multiplication by a test function $v \in V = H^1(\Omega)$ and integration over Ω, using Green's theorem, which gives,

$$(\nabla u, \nabla v) = (f, v), \tag{12.21}$$

since the boundary term vanishes by the Neumann boundary condition. Thus the variational forms (12.18) and (12.21) are similar, with the only difference being the choice of test and trial spaces.

However, it turns out that the variational problem (12.21) has no unique solution, since for any solution $u \in V$, also $v + C$ is a solution, with C a constant. To ensure a unique solution, we need an extra condition for the solution, for example, we may change the approximation space to

$$V = \{v \in H^1(\Omega) : \int_\Omega v(x)\, dx = 0\}. \tag{12.22}$$

Non homogeneous boundary conditions

We now state the variational formulation of Poisson equation with non homogeneous boundary conditions,

$$-\Delta u = f, \qquad x \in \Omega, \tag{12.23}$$
$$u(x) = g_D, \qquad x \in \Gamma_D, \tag{12.24}$$
$$\nabla u \cdot n = g_N, \qquad x \in \Gamma_N, \tag{12.25}$$

with $\Gamma = \Gamma_D \cup \Gamma_N$, which we obtain by multiplication by a test function $v \in V$, with

$$V = \{v \in H^1(\Omega) : v(x) = g_D(x),\, x \in \Gamma_D\}, \tag{12.26}$$

and integration over Ω, using Green's theorem, which gives,

$$(\nabla u, \nabla v) = (f, v) + (g_N, v)_{\Gamma_N}. \tag{12.27}$$

The Dirichlet boundary condition is enforced through the trial space, and is thus referred to as an *essential boundary condition*, whereas the Neumann boundary condition is enforced through the variational form, thus referred to as a *natural boundary condition*.

The finite element method

To compute approximate solutions to the Poisson equation, we can formulate a finite element method based on the variational formulation of the equation, replacing the Sobolev space V with a polynomial space V_h, constructed by a set of basis functions $\{\phi_i\}_{i=1}^M$, over a mesh $\mathcal{T}_h$, defined as a collection of elements $\{K_i\}_{i=1}^N$ and nodes $\{N_i\}_{i=1}^M$.

For the Poisson equation with homogeneous Dirichlet boundary conditions, the finite element method takes the form: Find $U \in V_h$, such that,

$$(\nabla U, \nabla v) = (f, v), \quad v \in V_h, \tag{12.28}$$

with $V_h \subset H_0^1(\Omega)$.

The variational form (12.28) corresponds to a linear system of equations $Ax = b$, with $a_{ij} = (\phi_j, \phi_i)$, $x_j = U(N_j)$, and $b_i = (f, \phi_i)$, with $\phi_i(x)$ the basis function associated with the node N_i.

12.3 Linear partial differential equations

The abstract problem

We express a linear partial differential equation as the abstract problem,

$$Lu = f, \quad x \in \Omega, \tag{12.29}$$

with boundary conditions,

$$Bu = g, \quad x \in \Gamma, \tag{12.30}$$

for which we can derive a variational formulation: find $u \in V$ such that,

$$a(u, v) = L(v), \quad v \in V, \tag{12.31}$$

with $a : V \times V \to \mathbb{R}$ a *bilinear form*, that is a function which is linear in both arguments, and $L : V \to \mathbb{R}$ a *linear form*.

In a Galerkin method we seek an approximation $U \in V_h$ such that

$$a(U, v) = L(v), \quad v \in V_h, \tag{12.32}$$

with $V_h \subset V$ a finite dimensional subspace, which in the case of a finite element method is a polynomial space.

Energy error estimation

A bilinear form $a(\cdot, \cdot)$ on the Hilbert space V is *symmetric*, if

$$a(v, w) = a(w, v), \quad v, w \in V, \tag{12.33}$$

and *coercive*, or *elliptic*, if

$$a(v, v) \geq c\|v\|, \quad v \in V, \tag{12.34}$$

with $c > 0$. A symmetric and elliptic bilinear form defines an inner product on V, which induces a norm which we refer to as the *energy norm*,

$$\|w\|_E = a(w, w)^{1/2}. \tag{12.35}$$

The Galerkin approximation is optimal in the norm, since by Galerkin orthogonality,

$$a(u - U, v) = 0, \quad v \in V_h, \tag{12.36}$$

we have that

$$\begin{aligned} \|u - U\|_E^2 &= a(u - U, u - u_h) = a(u - U, u - v) + a(u - U, v - u_h) \\ &= a(u - U, u - v) \leq \|u - U\|_E \|u - v\|_E, \end{aligned}$$

so that

$$\|u - U\|_E \leq \|u - v\|_E, \quad v \in V_h. \tag{12.37}$$