EP2200 Queueing theory and teletraffic systems

M/G/1 systems with vacation and priority

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M/G/1 queues

- Recall:
  - Arrival process: Poisson
  - Service process: i.i.d service times
    - the first and second moment determine the average performance measures in the queue
    - the distribution has to be known to derive distribution of performance metrics
- Pollaczek-Khinchin mean formulas
  - based on mean value analysis

$$W = \frac{R_s}{1-\rho} = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\rho E[X]}{2(1-\rho)}(1+C_x^2)$$

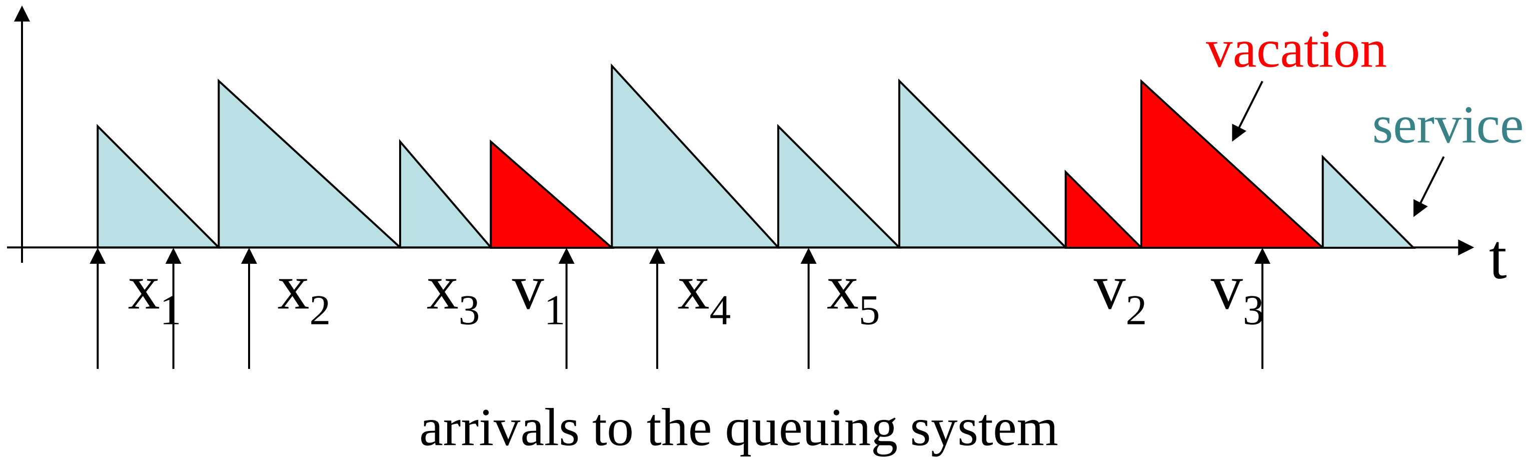
- Pollaczek-Khinchin transform equations
  - based on embedded MC

$$Q(z) = B^*(\lambda - \lambda z)\frac{(1-\rho)(1-z)}{B^*(\lambda-\lambda z)-z} \qquad T^*(s) = B^*(s)\frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)}$$

M/G/1 with vacation

- Vacation: the server is not available for a while after the system gets idle (empty)
  - there is no idle period, only vacation period
  - vacation periods: identically distributed, independent random variable, V
- Stability condition: $\lambda E[X]<1$ – higher load $\rightarrow$less vacation

$R_s(t)$, $R_v(t)$ {remaining service time, remaining vacation time}



M/G/1 with vacation – waiting time

$$E\left[W_{k|b}\right] = E\left[R_{s,k|b}\right] + (k-1)E\left[X\right], \quad k \geq 1 \quad \text{(waiting time for customer arriving when server is busy)}$$

$$E\left[W_{k|v}\right] = E\left[R_{v,k|v}\right] + kE\left[X\right], \quad k \geq 0 \quad \text{(waiting time for customer arriving when server is on vacation)}$$

$$\text{Def}: R_{s,k|v} = 0, \quad R_{v,k|b} = 0$$

$$E\left[W \mid b\right] = \sum_{k=0}^{\infty} p_{k|b} E\left[R_{s,k|b}\right] + \sum_{k=0}^{\infty} p_{k|b} (k-1) E\left[X\right]$$

$$E\left[W \mid v\right] = \sum_{k=0}^{\infty} p_{k|v} E\left[R_{v,k|b}\right] + \sum_{k=0}^{\infty} p_{k|v} k E\left[X\right]$$

$$W = (1-\rho)E\left[W \mid v\right] + \rho E\left[W \mid b\right]$$

R_s: average remaining service time, averaged over time including even vacation periods
R_v: average remaining vacation time, averaged over time including even service periods

Still, the system is busy with probability ρ.

M/G/1 with vacation – waiting time

$$E[W \mid b] = \sum_{k=0}^{\infty} p_{k|b} E[R_{s,k|b}] + \sum_{k=0}^{\infty} p_{k|b} (k-1) E[X]$$

$$E[W \mid v] = \sum_{k=0}^{\infty} p_{k|v} E[R_{v,k|b}] + \sum_{k=0}^{\infty} p_{k|v} k E[X]$$

$$W = (1-\rho) E[W \mid v] + \rho E[W \mid b]$$

R_s: average remaining service time, averaged over time including even vacation periods
R_v: average remaining vacation time, averaged over time including even service periods

$$R_s = (1-\rho)0 + \rho \sum_{k=0}^{\infty} p_{k|b} E[R_{s,k|b}], \quad R_v = (1-\rho) \sum_{k=0}^{\infty} p_{k|v} E[R_{v,k|v}] + \rho 0$$

$$(1-\rho) \sum_{k=0}^{\infty} p_{k|v} k E[X] + \rho \sum_{k=0}^{\infty} p_{k|b} (k-1) E[X] = ((1-\rho) N_{q|v} + \rho N_{q|b}) E[X] = N_q E[X]$$

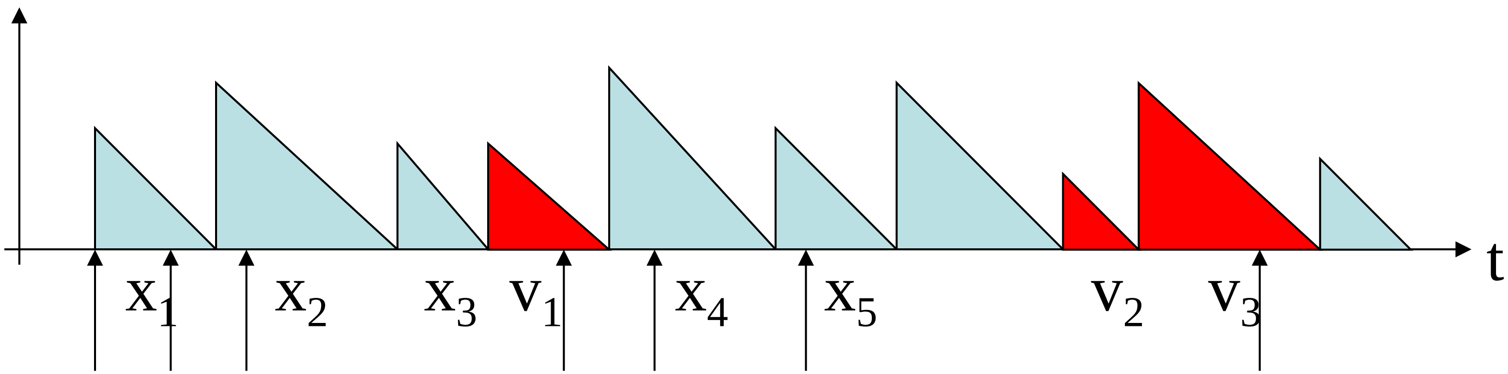
$$W = N_q E[X] + R_s + R_v$$

$$W = \lambda W E[X] + R_s + R_v$$

$$W(1-\rho) = R_s + R_v$$

$$W = \frac{R_s}{1-\rho} + \frac{R_v}{1-\rho} = \frac{\lambda E[X^2]}{2(1-\rho)} + \frac{R_v}{1-\rho}, \quad \left[\text{Recall M/G/1: } R_s = \frac{\lambda E[X^2]}{2} \right]$$

M/G/1 with vacation – waiting time



$$R_v = \frac{\sum_{i=1}^{n} \frac{1}{2} v_i^2}{T}, \quad \text{where} \quad T(1-\rho) = \sum_{i=1}^{n} v_i \quad \Rightarrow \quad \frac{1}{T} = \frac{(1-\rho)}{\sum_{i=1}^{n} v_i}$$

$$R_v = \frac{(1-\rho)\sum_{i=1}^{n} \frac{1}{2} v_i^2}{\sum_{i=1}^{n} v_i} = \frac{(1-\rho)}{2} \frac{\frac{1}{n}\sum_{i=1}^{n} v_i^2}{\frac{1}{n}\sum_{i=1}^{n} v_i} = \frac{(1-\rho)}{2} \frac{E[V^2]}{E[V]}$$

$$W = \frac{\lambda E[X^2]}{2(1-\rho)} + \frac{R_v}{1-\rho} = \frac{\lambda E[x^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]}$$

Calculate average remaining vacation time:

- consider the system for time T, within that vacation for (1-ρ)T
- n vacation periods, $n \to \infty$

The M/G/1 mean value analysis Pollaczek-Khinchin mean formulas

Group work:

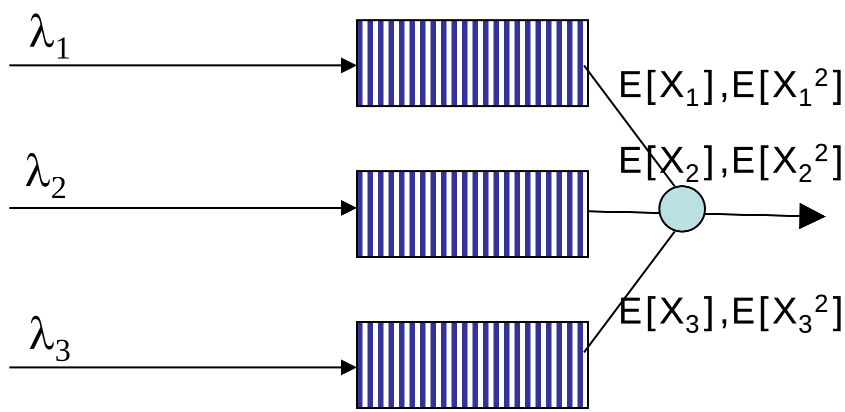
- Consider the following system:
  - Single server, infinite buffer
  - Poisson arrival process, 0.1 customer per minute
  - Service process: sum of two exponential steps, with mean times 1 minute and 2 minutes
  - Maintenance period starts whenever the system becomes idle, the maintenance takes exactly 1 minute.
  - Calculate the mean waiting time

$$W = \frac{\lambda E[x^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]}$$

(Similar to M/G/1 without vacation problem, where we had W=1min)

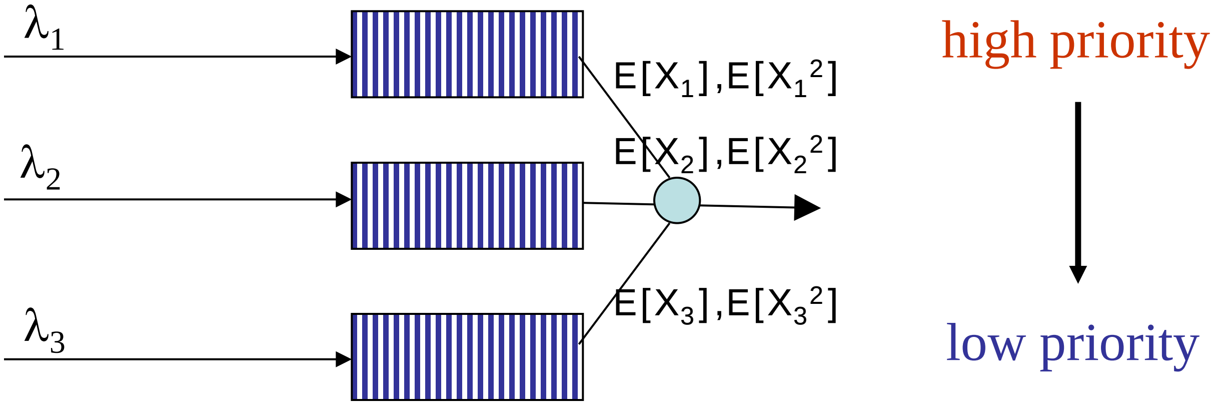
M/G/1 with priority

- M/G/1 queue
- K priority class:
  - Separate, infinite queue for each class, one server (multiclass system)
  - Poisson arrival in each class ($\lambda_{i,}$ $\sum \lambda_i = \lambda$)
  - General service time distribution in each class ($E[X_i], E[X_i^2]$)
    - Service time distribution looks like the linear combination of distributions with probabilities λ_i / λ
      - $E[X] = \sum \lambda_i / \lambda \, E[X_i]$
      - $E[X^2] = \sum \lambda_i / \lambda \, E[X_i^2]$,
  - Class 1 the highest priority



M/G/1 with priority

- Priority systems
  - Low priority customer selected only if high priority queues are empty
  - Non-preemptive: the service is completed even if higher priority customer arrives
  - Preemptive: the service is interrupted if higher priority customer arrives
    - Resume: the service continues from the point of interruption
    - Non-resume: the service starts from the beginning (not considered in this course)



M/G/1 with non-preemptive priority

- Derive mean performance parameters
- Waiting time for a customer of priority *i* =

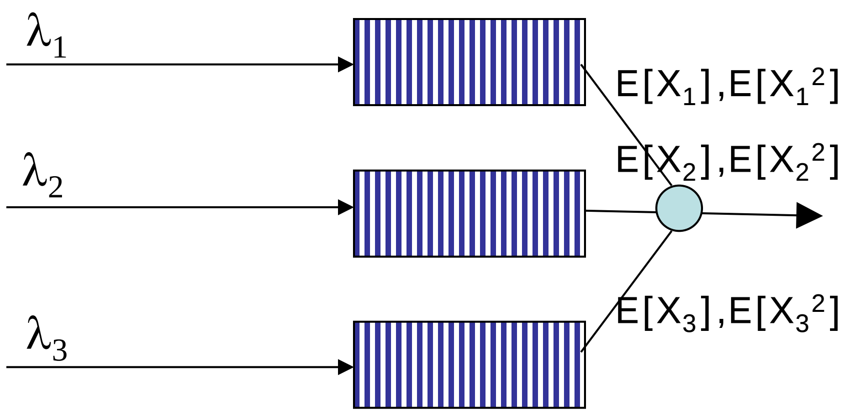
  Residual service time, R_s +

  Service time of customers already waiting in queue *i* +

  Service time of customers already waiting in queues *j*<*i* (higher priority) +

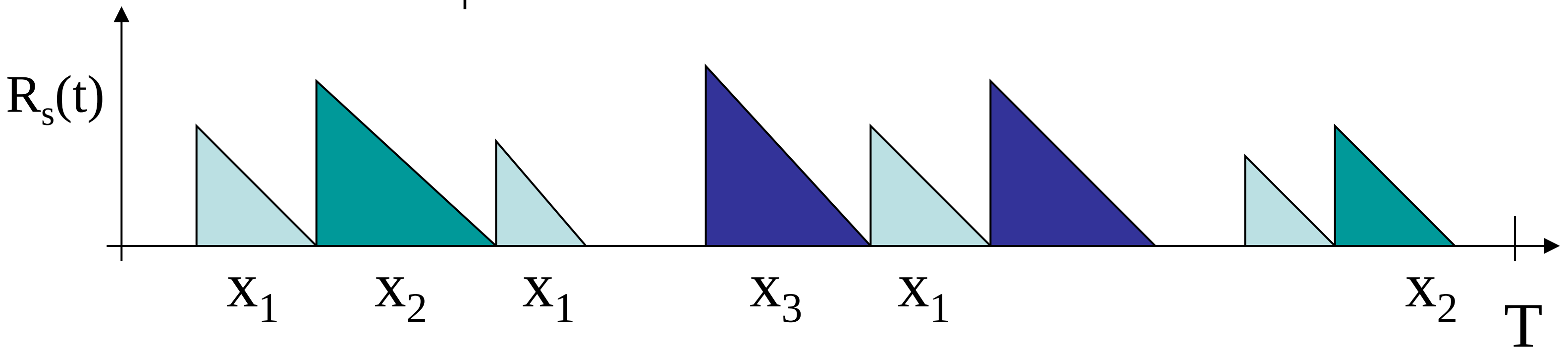
  Service time of customers arriving to queues j<i while "our" customer is waiting

$$W_i = R_s + E[X_i]N_{q,i} + \sum_{j=1}^{i-1} E[X_j]N_{q,j} + \sum_{j=1}^{i-1} E[X_j]\lambda_j W_i$$



The M/G/1 mean value analysis Pollaczek-Khinchin mean formulas

- We have to derive the average remaining service time:
  - n: number of services in a large T = number of Poisson arrivals: $n_i=\lambda_i T$
  - $T \to \infty$ and $n_i \to \infty$



$$R_s = E[R_s(t)] = \frac{1}{T}\sum_{i=1}^{K}\sum_{j=1}^{n_i}\frac{1}{2}X_{i,j}^2 = \frac{1}{2}\sum_{i=1}^{K}\frac{\lambda_i}{n_i}\sum_{j=1}^{n_i}X_{i,j}^2 = \boxed{\frac{1}{2}\sum_{i=1}^{K}\lambda_i E[X_i^2]}$$

$$W_i = R_s + E[X_i]N_{q,i} + \sum_{j=1}^{i-1}E[X_j]N_{q,j} + \sum_{j=1}^{i-1}E[X_j]\lambda_j W_i$$

- Express W_1 and W_2!

M/G/1 with non-preemptive priority

- W_i , T_i general form:

$$W_i = \frac{R_s}{(1-\sum_{j=1}^{i-1}\rho_j)(1-\sum_{j=1}^{i}\rho_j)}, \quad R_s = \frac{1}{2}\sum_{i=1}^{K}\lambda_i E[X_i^{\,2}]$$

$$T_i = W_i + E[X_i]$$

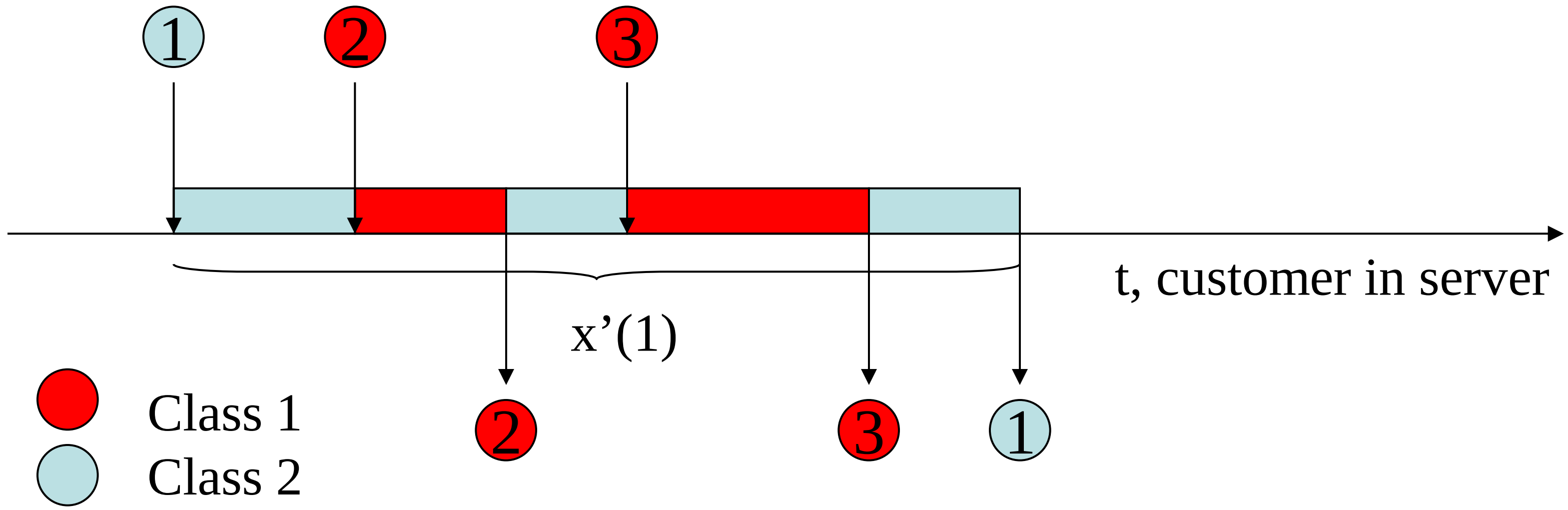
- Average waiting time:

$$W = \sum p_i W_i = \sum \frac{\lambda_i}{\lambda} W_i$$

- Comments:
  - W_i depends on X_j even if i<j (through the residual service time)
  - Mean waiting time W can be decreased if shorter service gets priority
    - in multiclass systems average perf. measures are dependent on the service policy

M/G/1 with preemptive resume priority

- Service is interrupted if higher priority customer arrives
  – later the service continues from the point of interruption
- Derive mean performance parameters
- Now: lower class customer is invisible for higher class customers!
- Definition of service time of low priority customers (x’):
  – From the time of first entering the server until competition.



M/G/1 with preemptive resume priority

- Waiting time for a customer of priority *i*

  = Time from arrival to the first service attempt =

  residual service time, $R_{s,i}$ (now priority dependent) +

  service time of customers already waiting in queue *i* +

  service time of customers already waiting in queues *j<i* (higher priority) +

  service time of customers arriving to queues j<i while "our" customer is waiting

$$R_{s,i} = \frac{1}{2}\sum_{j=1}^{i} \lambda_j E[X_j^2]$$

$$W_i = \frac{R_{s,i}}{(1-\sum_{j=1}^{i-1}\rho_j)(1-\sum_{j=1}^{i}\rho_j)}$$

Recall: non-preemptive case:

$$R_s = \frac{1}{2}\sum_{i=1}^{K} \lambda_i E[X_i^2]$$

  - E.g., highest priority (class 1) customer?

- Comment: now the high priority customers do not "see" the lower priority ones!

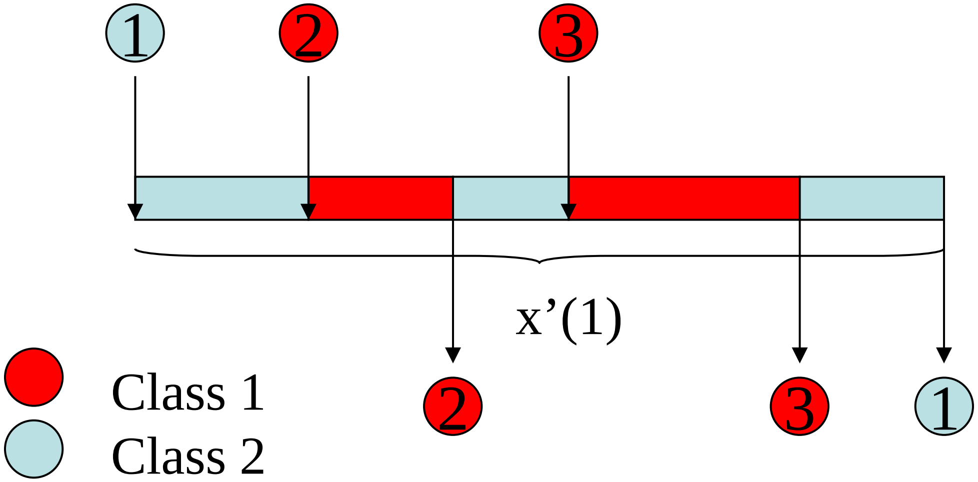
M/G/1 with preemptive resume priority

- Mean system time (T_i)?
  - Waiting time + service time including interruptions by arriving high priority customers

$$E[X'_i] = E[X_i] + \sum_{j=1}^{i-1} E[X_j]\lambda_j E[X'_i]$$

$$E[X'_i] = \frac{E[X_i]}{1-\sum_{j=1}^{i-1}\rho_j}$$

$$T_i = \frac{R_{s,i}}{(1-\sum_{j=1}^{i-1}\rho_j)(1-\sum_{j=1}^{i}\rho_j)} + \frac{E[X_i]}{1-\sum_{j=1}^{i-1}\rho_j}$$



  - E.g., average service time for class 1 and for class 2 customer?

M/G/1 with vacation and with priorities

- Requirements for the exam:

- Derive and use P-K mean value formulas
- Understand the concept of remaining vacation time
- Understand the concept of remaining service time in priority systems
- Calculate expected remaining vacation and service times with different conditions (for all customers, for customers finding the system empty/busy)
- Understand the concept of service time in the preemptive priority system, calculate it for specific cases

- Exam: formula sheet is available.