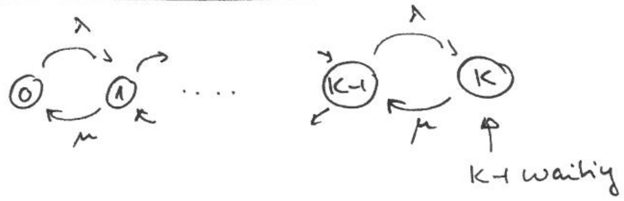
M/M/1/K



$\rho = \frac{\lambda}{\mu}$ (the correct notation would be $a = \frac{\lambda}{\mu}$)

Stability: always stable (finite, irreducible)

$$\left.\begin{aligned} \lambda p_0 &= \mu p_1 \\ \lambda p_1 &= \mu p_2 \\ &\vdots \\ \lambda p_{K-1} &= \mu p_K \\ \sum_{k=0}^{K} p_k &= 1 \end{aligned}\right\} \quad p_k = p_0 \cdot \rho^k \qquad p_0 = \left[1 + \rho + \dots + \rho^K\right]^{-1}$$

- $\rho = 1 \Rightarrow p_k = \frac{1}{K+1}$
- $\rho \neq 1 \quad p_0 = \frac{1}{\sum_{k=0}^{K} \rho^k}$

$$\sum_{k=0}^{K} \rho^k = \rho^0 + \rho^1 + \rho^2 \dots \rho^K$$

$$(1-\rho)\sum_{k=0}^{K} \rho^k = (1-\rho)(\rho^0 + \rho^1 \dots) = \rho^0 + \rho^1 + \dots \rho^K - \rho^1 - \rho^2 \dots - \rho^{K+1}$$

$$\sum_{k=0}^{K} \rho^k = \frac{1-\rho^{K+1}}{1-\rho} \Rightarrow p_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

Blocking prob.:

$$P(\text{block}) = a_K = p_K = \frac{(1-\rho)\rho^K}{1-\rho^{K+1}}$$

Effective traffic, utilization

$$\lambda_{eff} = (1-p_K)\lambda$$

$$u = \lambda_{eff} \cdot \frac{1}{\mu} \qquad [= 1 - p_0]$$

~~Perf~~ Perf. metrics

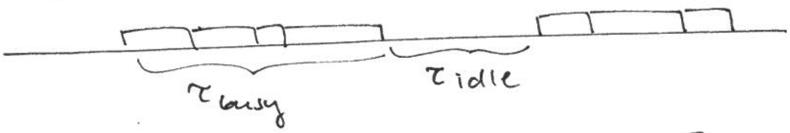
$$\bar{N} = \sum_{k=0}^{K} k p_k = \frac{\rho}{1-\rho}\left(1 - (K+1)p_K\right), \quad N_S = \lambda_{eff}\bar{x} \dots$$

Length of idle, blocking and busy period

$\bar{\tau}_{idle} = \frac{1}{\lambda} \qquad \tau_{idle} \sim Exp(\lambda)$

$\bar{\tau}_{block} = \frac{1}{\mu} \qquad \tau_{block} \sim Exp(\mu)$

$\bar{\tau}_{busy}$?



$$u = \frac{\bar{\tau}_{busy}}{\bar{\tau}_{busy} + \bar{\tau}_{idle}} \Rightarrow \bar{\tau}_{busy} = \frac{u \cdot \bar{\tau}_{idle}}{1-u}$$

Example: M/M/1/2, $\rho = 1$, $\lambda = \mu = 1$

$p_0 = p_1 = p_2 = \frac{1}{3}$

$P(\text{block}) = \frac{1}{3}$

$\lambda_{eff} = \frac{2}{3}, \quad u = \frac{2}{3}$

$\bar{N} = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$

$\bar{\tau}_{idle} = \frac{1}{\lambda} = 1$

$\bar{\tau}_{block} = \frac{1}{\mu} = 1$

$\tau_{busy} = \frac{2/3 \cdot 1}{1 - \frac{2}{3}} = 2$

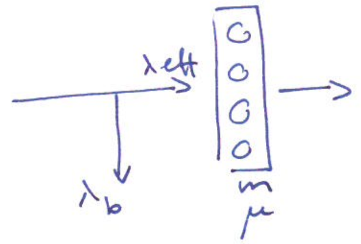
M/M/m/m : Erlang ~~block~~ loss system)

Erlang (Danish) 1878–1929

1. System definition

- Arrival : Poisson (λ)
- Service : Exp(μ), $\bar{x} = \frac{1}{\mu}$
- m servers (selected randomly)
- no buffer

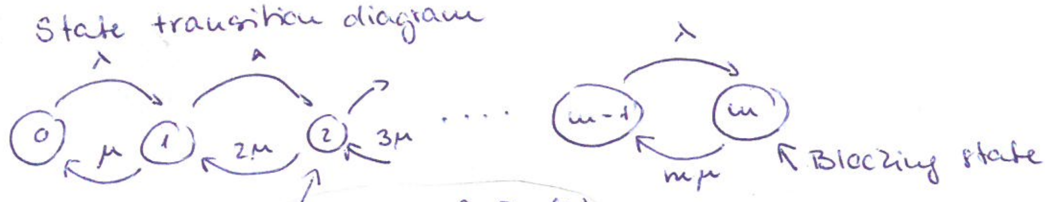
Offered load
$a = \lambda \bar{x} = \frac{\lambda}{\mu}$ [Erlang]



Steady state always exist

State: # customers in the system = # occupied servers

~~3~~ State transition diagram



Minimum of 2 Exp(μ)
=> Exp(2μ)

$P(\tau < t) = P(\text{at least one service}) = 1 - P(\text{no service})$
$1 - \prod P(\text{server } i \text{ has not finished}) =$
$1 - \prod e^{-\mu t} = 1 - e^{-\sum \mu i t} \Rightarrow \text{Exp}(i\mu)$

2. ~~Data~~ Steady state prob. p

(group work)

$p_0 \lambda = p_1 \mu$
$p_1 \lambda = p_2 \cdot 2\mu$
$p_2 \lambda = p_3 \cdot 3\mu$
$\vdots$
$p_{m-1} \lambda = p_m m\mu$
$\sum_{k=0}^{m} p_k = 1$

$p_1 = \frac{\lambda}{\mu} p_0 = a \cdot p_0$
$p_2 = \frac{\lambda}{2\mu} p_1 = \frac{a^2}{2} p_0$
$p_3 = \frac{\lambda}{3\mu} p_2 = \frac{a^3}{3!} p_0$
$\vdots$
$p_m = \frac{a^m}{m!} p_0$

$\sum_{k=0}^{m} \frac{a^k}{k!} p_0 = 1 \Rightarrow p_0 = \frac{1}{\sum_{k=0}^{m} \frac{a^k}{k!}}$

$$\boxed{p_k = \frac{a^k/k!}{\sum_{i=0}^{m} a^i/i!}}$$

Note: $m \to \infty$ $\quad p_k = \frac{a^k/k!}{e^a}$

$\Leftarrow$ truncated Poisson

3. Performance measures

$$P(\text{blocking}) = \overset{P(\text{arriving}) = \text{PASTA}}{p_m} = \frac{a^m/m!}{\sum_i a^i/i!} = E_m(a) = B(m,a) \quad \boxed{\text{Erlang-B form}}$$

$W = 0, \quad N_q = 0$

$T = \bar{x} = \frac{1}{\mu}, \quad N = N_S \overset{\text{Little}}{=} \lambda_{eff} \bar{x} = (1-p_m)\lambda\bar{x} = (1 - p_m) \cdot a, \quad$ utilization: $\rho = \frac{\lambda_{eff} \cdot \bar{x}}{m} = (1-p_m) \cdot \frac{a}{m}$ (< 1)

4. Erlang table

$a = A = 3$, $p(\text{block}) < 0.1 \Rightarrow m = 6$

$a = A = 6$, $p(\text{block}) < 0.1 \Rightarrow m = 9$

$\uparrow$ e.g. for double λ

$\Rightarrow$ efficiency of multiplexing

M/M/∞/∞