Recitation 5

Chapter 5:

$\boxed{5.1}$

$\boxed{5.2}$

$\boxed{5.5}$

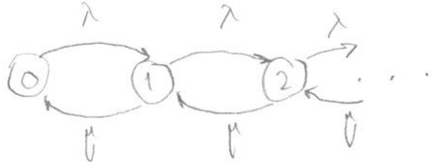
M/M/1/∞

- Exp. interarrival times (λ)
- Exp. service times (μ)

$\Big\}$ independent $\Big\}$ $\rho = \frac{\lambda}{\mu}$ ← utilization of the system

- 1 server
- ∞ buffer capacity

$a = \lambda \cdot \bar{x}$ - offered load



stability $\rho < 1$

$$P_k = (1-\rho)\cdot \rho^k$$

$$\bar{N} = \frac{\rho}{1-\rho} \Rightarrow \bar{T} = \frac{\bar{N}}{\lambda} = \frac{1}{\mu - \lambda}$$

N_q, N_s, W, $\bar{x}$ from Littles

$T(t) = P(T \leq t) = 1 - e^{-(\mu-\lambda)\cdot t}$ - system time distribution for FIFO system

$W(t) = 1 - \rho \cdot e^{-(\mu-\lambda)\cdot t}$ - waiting time distribution for FIFO system

$$\sum_{k=1}^{\infty} \rho^k = \frac{\rho}{1-\rho}$$ ← infinite geomethric series

$$\sum_{k=0}^{\infty} \frac{\rho^k}{k!} = e^{\rho}$$

z-transform def:

$$G_x(z) = \sum_{i=0}^{\infty} p_i \cdot z^i$$

Moments with the help of z-transform

$$E[X] = \frac{d}{dz} G(z)\Big|_{z=1}$$

$$E[X^2] = \frac{d^2}{dz^2} G(z)\Big|_{z=1} + \frac{d}{dz} G(z)\Big|_{z=1}$$

Laplace-transform def:

$$f_x^*(s) = \int_0^{\infty} e^{-sx} \cdot f(x)\, dx$$

Moments with the help of Laplace-trans:

$$E[X] = \frac{-d}{ds} f_x^*(s)\Big|_{s=0}$$

$$E[X^2] = \frac{d^2}{ds^2} f^*(s)\Big|_{s=0}$$

J.1) Arrivals: Poisson(λ)

Length of the messages: $L \sim Exp(\mu_L)$

FCFS (FIFO)

M/M/1

$\frac{1}{\mu_L}$ - mean length of the messages

Transm rate C $\frac{bits}{s}$

$\frac{1}{\mu}$ - mean service time

Service: $Exp(C \cdot \mu_L)$

$$\boxed{\frac{1}{\mu} = \frac{1/\mu_L}{C} = \frac{1}{\mu_L C}}$$

1.

a.) $C = ?$

$$\bar{T} = \frac{1}{\mu - \lambda} = \frac{1}{\mu_L \cdot C - \lambda}$$

$$\bar{T} < T_0$$

$$\frac{1}{\mu_L C - \lambda} < T_0$$

$$\frac{1}{T_0} < \mu_L \cdot C - \lambda \quad (*)$$

$$\boxed{C > \frac{\frac{1}{T_0} + \lambda}{\mu_L}}$$

$$P(T > 3T_0) \overset{CCDF}{=} 1 - P(T \leq 3T_0) = 1 - T(3T_0) = 1 - 1 + e^{-\underbrace{(C \cdot \mu_L - \lambda)}_{\frac{1}{T_0}\ (*)} \cdot 3T_0} =$$

system time distribution for FIFO system

$$= e^{-\frac{1}{T_0} \cdot 3T_0} = e^{-3}$$

b.) $C = ?$

$$P(T > t) < p$$

$$e^{-(C\mu_L - \lambda) \cdot t} < p$$

$$-(C \cdot \mu_L - \lambda) \cdot t < \ln(p)$$

$$-C \mu_L \cdot t + \lambda t < \ln(p)$$

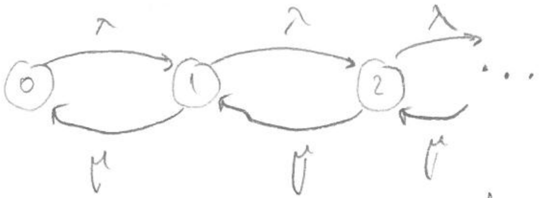
$$\boxed{C > \frac{-\frac{\ln(p)}{t} + \lambda}{\mu_L}}$$

c.) $\lambda = ?$

$$P(T > t) < p$$

$$\boxed{\lambda < \frac{\ln(p)}{t} + C \mu_L}$$

5.2) M/M/1



$\rho = \frac{\lambda}{\mu}$

a.) $T = \sum_{k=0}^{\infty} E[T \mid k) \cdot p_k = \sum_{k=0}^{\infty} \frac{k+1}{\mu} \cdot p_k = \frac{1}{\mu} \cdot \left(\underbrace{\sum_{k=0}^{\infty} k \cdot p_k} + \sum_{k=0}^{\infty} p_k \right) =$

$= \frac{1}{\mu} \cdot \left(\bar{N} + 1 \right) = \frac{1}{\mu} \left(\frac{\rho}{1-\rho} + 1 \right) = \frac{1}{\mu - \lambda}$

b.) LCFS

$T =$ P(empty system) $\cdot T_{\text{own service}}$ +

P(non-empty system) $\cdot$ [$T_{\text{ongoing-service}}$ + $T_{\text{service of jobs arriving while ongoing}}$ + $\underbrace{\text{service after me}}$

+ $T_{\text{service of jobs arriving while}}$ service +

+ $T_{\text{own service}}$] =

$= p_0 \cdot \frac{1}{\mu} + (1-p_0) \cdot \left[\frac{1}{\mu} + \left(\lambda \cdot \frac{1}{\mu} \right) \frac{1}{\mu} + \lambda \cdot \left(\lambda \cdot \frac{1}{\mu} \frac{1}{\mu} \right) \frac{1}{\mu} + \ldots + \frac{1}{\mu} \right) =$

for each of new $\frac{1}{\mu}$ is service

$= p_0 \cdot \frac{1}{\mu} + (1-p_0) \cdot \left(\sum_{i=0}^{\infty} \left(\lambda \cdot \frac{1}{\mu} \right)^i \frac{1}{\mu} + \frac{1}{\mu} \right) =$

$= p_0 \cdot \frac{1}{\mu} + (1-p_0) \cdot \left[\frac{1}{1 - \lambda \cdot \frac{1}{\mu}} \cdot \frac{1}{\mu} + \frac{1}{\mu} \right] = \left(1 - \frac{\lambda}{\mu} \right) \cdot \frac{1}{\mu} +$

$p_0 = 1 - \frac{\lambda}{\mu}$

$+ \left(\frac{\lambda}{\mu} \right) \cdot \left(\frac{1}{1 - \frac{\lambda}{\mu}} \cdot \frac{1}{\mu} + \frac{1}{\mu} \right) =$

$= \frac{1}{\mu} - \cancel{\frac{\lambda}{\mu^2}} + \frac{\lambda}{\mu} \cdot \frac{1}{1 - \frac{\lambda}{\mu}} \cdot \frac{1}{\mu} + \cancel{\frac{\lambda}{\mu^2}} = \frac{1}{\mu} + \frac{\lambda}{\mu^2} \cdot \frac{\cancel{\mu}}{\mu - \lambda} =$

$= \frac{1}{\mu} + \frac{\lambda}{\mu \cdot (\mu - \lambda)} = \frac{\mu - \cancel{\lambda} + \cancel{\lambda}}{\mu \cdot (\mu - \lambda)} = \frac{1}{\mu - \lambda}$

c.) $T = \dfrac{N}{\lambda} = \dfrac{\sum_{k=0}^{\infty} k \cdot p_k}{\lambda} = \dfrac{\frac{\lambda/\mu}{1-\lambda/\mu}}{\lambda} = \dfrac{1}{\mu \cdot \left(1 - \frac{\lambda}{\mu}\right)}$

d.) $E[N^2] = \sum_{k=0}^{\infty} k^2 \cdot p_k = \sum_{k=0}^{\infty} k^2 \cdot (1-\rho) \cdot \rho^k = (1-\rho) \cdot \sum_{k=0}^{\infty} k^2 \cdot \rho^k =$

there is no contribution for $k=0$

$$= (1-\rho)\rho^2 \sum_{k=1}^{\infty} k^2 \cdot \rho^{k-2} = \underbrace{(1-\rho) \cdot \rho^2 \cdot \sum_{k=1}^{\infty} k^2 \cdot \rho^{k-2} - (1-\rho) \cdot \rho^2 \cdot \sum_{k=1}^{\infty} k \cdot \rho^{k-2}} + (1-\rho) \cdot \rho^2 \cdot \sum_{k=1}^{\infty} k \cdot \rho^{k-2} =$$

$$= (1-\rho) \cdot \rho^2 \cdot \sum_{k=1}^{\infty} \left(\rho^k\right)'' + (1-\rho) \cdot \rho \sum_{k=1}^{\infty} \left(\rho^k\right)' =$$

$$= (1-\rho) \cdot \rho^2 \cdot \left(\sum_{k=1}^{\infty} \rho^k\right)'' + (1-\rho) \cdot \rho \cdot \left(\sum_{k=1}^{\infty} \rho^k\right)' =$$

$$= (1-\rho) \cdot \rho^2 \cdot \left(\frac{\rho}{1-\rho}\right)'' + (1-\rho) \cdot \rho \cdot \left(\frac{\rho}{1-\rho}\right)' =$$

$$= (1-\rho) \cdot \rho^2 \left(\frac{1 \cdot (1-\rho) - (-1) \cdot \rho}{(1-\rho)^2}\right)' + (1-\rho) \cdot \rho \cdot \frac{1 \cdot (1-\rho) - (-1) \cdot \rho}{(1-\rho)^2} =$$

$$= \cancel{(1-\rho)} \cdot \rho^2 \ \frac{0 \cdot (1-\rho)^2 - 2 \cdot (\cancel{1-\rho}) \cdot (-1) \cdot 1}{(1-\rho)^{\cancel{4}}} + (1-\rho) \cdot \rho \cdot \frac{1}{(1-\rho)^2} =$$

$$= \rho^2 \cdot \frac{2}{(1-\rho)^2} + (1-\rho) \cdot \rho \cdot \frac{1}{(1-\rho)^2} =$$

$$= \frac{2\rho^2 + \rho - \rho^2}{(1-\rho)^2} = \frac{\rho^2 + \rho}{(1-\rho)^2} = \frac{\rho \cdot (1+\rho)}{(1-\rho)^2}$$

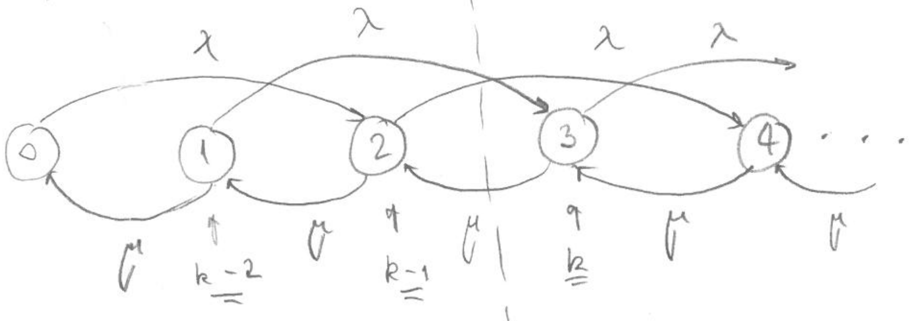
$$Var[N] = E[N^2] - E[N]^2 = \frac{\rho \cdot (1+\rho)}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho)^2} = \frac{\rho}{(1-\rho)^2}$$

5.5
- single server
- arrivals - Poisson dist.
- 2 customers per arrival
- service times - Exp. dist.

1.

M/M/1

a.)



b.) $\lambda \cdot p_0 = \mu \cdot p_1$; $\lambda \cdot p_{k-2} + \lambda \cdot p_{k-1} = \mu \cdot p_k$, $k \geq 1$ $p_{-1} = 0$

c.) $\lambda \cdot p_{k-2} + \lambda \cdot p_{k-1} = \mu \cdot p_k \quad | \cdot z^k$, add up from $k = 2$

$$\sum_{k=2}^{\infty} z^k \cdot (\lambda \cdot p_{k-2} + \lambda \cdot p_{k-1}) = \sum_{k=2}^{\infty} z^k \mu \cdot p_k$$ / try to find $\boxed{P(z) = \sum_{i=0}^{\infty} z^i \cdot p_i}$

$j = k-2 \Rightarrow k = j+2$

$$\lambda \cdot \left(\sum_{j=0}^{\infty} z^{j+2} \cdot (p_j + p_{j+1}) \right) = \mu \cdot \left(\sum_{j=0}^{\infty} z^{j+2} \cdot p_{j+2} \right)$$

$i = j+1$

$$\lambda \cdot \left(z^2 \cdot \sum_{j=0}^{\infty} z^j \cdot p_j + \sum_{i=1}^{\infty} z^{i+1} \cdot p_i \right) = \mu \cdot \left(\sum_{i=2}^{\infty} z^i \cdot p_i \right)$$

$i = j+2$

$$\lambda \cdot \left(z^2 \cdot \sum_{j=0}^{\infty} z^j p_j + z \cdot \left(\sum_{i=0}^{\infty} z^i \cdot p_i - p_0 \right) \right) = \mu \cdot \left(\sum_{i=0}^{\infty} z^i \cdot p_i - z p_1 - p_0 \right)$$

$$\lambda \cdot z \cdot (z P(z) + P(z) - p_0) = \mu \cdot (P(z) - z p_1 - p_0)$$

$$P(z) \cdot (\lambda \cdot z^2 + \lambda z - \mu) = \lambda z p_0 - \mu z p_1 - \mu p_0 \quad / \cdot \left(-\frac{1}{\mu}\right)$$

$$P(z) = \frac{p_0 + z p_1 - \rho z p_0}{1 - \rho \cdot z - \rho \cdot z^2}$$

$p_0 = ?$
$p_1 = ?$

(*) $\Rightarrow p_1 = \rho \cdot p_0$

(2.)

$$\Rightarrow P(z) = \frac{p_0}{1 - \rho z - \rho z^2}$$

$$\sum_{k=0}^{\infty} p_k \cdot z^k \Big|_{z=1} = \sum_{k=0}^{\infty} p_k = 1 \Rightarrow$$

$$1 = \frac{p_0}{1-\rho-\rho} \Rightarrow p_0 = 1 - 2\rho \Rightarrow$$

$$P(z) = \frac{1-2\rho}{1-\rho z - \rho z^2}$$

d.) $$\bar{N} = \frac{dP(z)}{dz}\Big|_{z=1} = \frac{0 - (-\rho - 2\rho)\cdot(1-2\rho z)}{(1-\rho z - \rho z^2)^2}\Big|_{z=1} =$$

↑ (z-transform)

$$= \frac{-1\cdot(-3\rho)\cdot\cancel{(1-2\rho)}}{(1-2\rho)^{\cancel{2}}} = \frac{3\rho}{1-2\rho} = \frac{3\lambda}{\mu - 2\lambda}$$

simple M/M/1: $\bar{N} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$