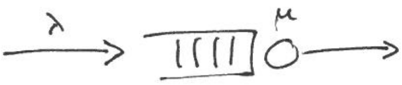
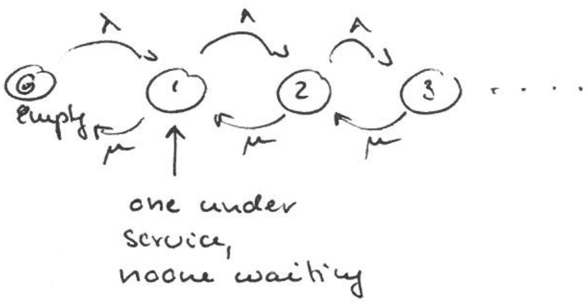
M/M/1

State: # customers in the system (server + queue)





Offered load $= a = \lambda \bar{x} = \frac{\lambda}{\mu}$

Utilization: $\rho = \frac{a}{m} = \frac{\lambda}{\mu}$

Stability: $\rho = \frac{\lambda}{\mu} < 1$

Stationary state probabilities

(as in the B-D lecture)

Local balance eq:

$\lambda p_0 = \mu p_1$ $\qquad p_1 = \frac{\lambda}{\mu} p_0$

$\lambda p_1 = \mu p_2$ $\qquad p_2 = \frac{\lambda}{\mu} p_1 = \frac{\lambda^2}{\mu^2} p_0$

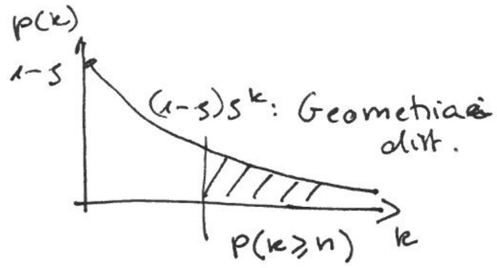
$\lambda p_2 = \mu p_3$

$\vdots$

$$p_k = \left(\frac{\lambda}{\mu}\right)^k p_0 = (1-\rho)\rho^k$$

$\sum_i p_i = 1$

$$p_0 = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k} = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

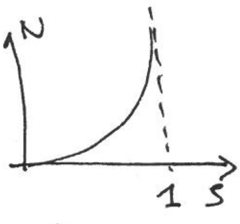


Average number of customers in the system

From the definition of the "state"

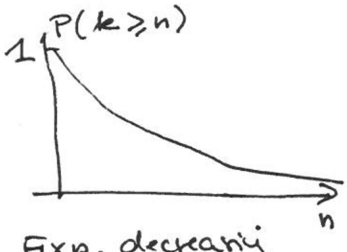
$$N = \sum_{k=0}^{\infty} k \cdot p_k = \sum_{k=1}^{\infty} k(1-\rho)\rho^k = (1-\rho)\rho \sum_{k=1}^{\infty} \underbrace{k \cdot \rho^{k-1}}_{(\rho^k)'} = (1-\rho)\rho \left(\sum_{k=1}^{\infty} \rho^k\right)' =$$

$$= (1-\rho)\cdot\rho\cdot\left(\frac{\rho}{1-\rho}\right)' = (1-\rho)\cdot\rho\cdot\frac{\overbrace{1\cdot(1-\rho) - \rho(-1)}^{1}}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$



Tail probability – at least n customers

$$P(k \geq n) = \sum_{k=n}^{\infty} p_k = \sum_{k=n}^{\infty} (1-\rho)\rho^k = (1-\rho)\frac{\rho^n}{1-\rho} = \rho^n$$



Exp. decreasing tail.

M/M/1 con't

Stationary mean perf. metrics

T, W, N_q, N_S Use Little: $N = \lambda T$
$N_S = \lambda \bar{x}$
$N_q = \lambda W$

$$T \overset{\text{Little}}{=} \frac{N}{\lambda} = \frac{\rho}{1-\rho} \cdot \frac{1}{\lambda} = \frac{1/\mu}{1-\rho} = \frac{1}{\mu - \lambda}$$

$$W = T - \bar{x} = \frac{1}{\mu-\lambda} - \frac{1}{\mu}$$

$$N_q = \lambda \cdot W \quad \text{or} \quad N_q = \sum_{k=1}^{\infty} (k-1) p_k$$

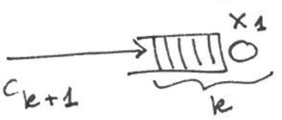
$$N_S = \lambda x \quad \text{or} \quad N_S = N - N_q \quad \text{or} \quad N_S = 0 \cdot p_0 + 1 \cdot (1 - p_0)$$

Scheduling discipline (FIFO, LIFO, Prio...)

- p_k does not depend on the scheduling
- average measures do not depend on it either!

System time distribution

$T(t) = P(\text{system time} \leq t)$, $f_T(t) = \frac{dT(t)}{dt}$



X_i: r.v., service time of request i

$T_k = \underbrace{X_1 + X_2 + \cdots}_{\text{waits}} \underbrace{X_{k+1}}_{\text{own service}}$

Laplace transform
$Exp(\mu) \Leftrightarrow \frac{\mu}{s+\mu}$
convolution $\Leftrightarrow$ product

$$\mathcal{L}(f_T(t|k)) = \prod_{i=1}^{k+1} \mathcal{L}(f(x)) = \left(\frac{\mu}{s+\mu}\right)^{k+1}$$

a_k: arriving customer finds the system in state k
p_k: stationary prob. that the system is in state k

PASTA: $a_k = p_k$

$$\mathcal{L}(f_T(t)) = \sum_{k=0}^{\infty} \mathcal{L}(f_T(t|k)) p_k = \sum_{k=0}^{\infty} \left(\frac{\mu}{s+\mu}\right)^{k+1} (1-\rho)\rho^k = (1-\rho)\frac{\mu}{s+\mu} \sum_{k=0}^{\infty} \left(\frac{\overbrace{\mu \cdot \rho}^{\lambda}}{s+\mu}\right)^k =$$

$$= (1-\rho)\frac{\mu}{s+\mu} \cdot \frac{1}{1-\frac{\lambda}{s+\mu}} = (1-\rho)\frac{\mu}{s+\mu} \frac{s+\mu}{s+\mu-\lambda} = \frac{\mu-\lambda}{s+(\mu-\lambda)} \Longleftrightarrow \underline{\underline{Exp(\mu-\lambda)}}$$

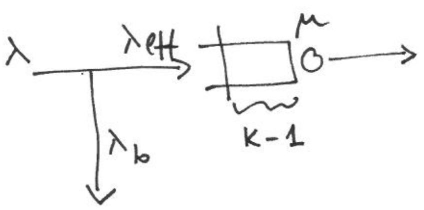
$\Rightarrow T \sim Exp(\mu-\lambda)$ $T(t) = 1 - e^{-(\mu-\lambda)t}$, $f_T(t) = (\mu-\lambda) e^{-(\mu-\lambda)t}$

Waiting time distribution

$$W(t) = 1 - \rho e^{-(\mu-\lambda)t}$$

M/M/1/K

1. Block diagram, Markov chain



$\Longrightarrow$

Offered load: $\varrho = \frac{\lambda}{\mu}$

Effective load: $\varrho_{eff} = \frac{\lambda_{eff}}{\mu} = \frac{(1 - P(\text{block}))\,\lambda}{\mu}$

2. Steady state, stationary state prob. (always stable: finite state, irreducible)

$$\left.\begin{aligned} &p_k = p_0 \cdot \varrho^k \quad \text{(like for M/M/1)} \\ &\sum_{k=0}^{K} p_k = 1 \end{aligned}\right\} \quad \begin{aligned} \sum_{k=0}^{K} p_k &= p_0\left(\sum_{k=0}^{\infty} \varrho^k - \sum_{k=K+1}^{\infty} \varrho^k\right) = \\ &= p_0\left(\frac{1}{1-\varrho} - \frac{\varrho^{K+1}}{1-\varrho}\right) = p_0\,\frac{1-\varrho^{K+1}}{1-\varrho} = 1 \end{aligned}$$

$$\Longrightarrow p_0 = \frac{1-\varrho}{1-\varrho^{K+1}}, \quad p_k = \frac{(1-\varrho)\,\varrho^k}{1-\varrho^{K+1}},$$

3. Performance

- Blocking probability

  P(arriving customer is blocked) = P(arrival to state K) $\overset{\text{PASTA}}{=} p_K = \frac{(1-\varrho)\,\varrho^K}{1-\varrho^{K+1}}$

- Effective traffic: $\lambda_{eff} = (1 - p_K)\lambda$
- Utilization $= \frac{\lambda_{eff}}{\mu}$
- $\bar{N} = \sum_{k=0}^{K} k \cdot p_k = \dots = \frac{\varrho}{1-\varrho}\left(1 - (K+1)p_K\right)$
- Other measures through Little's result.