Recitation 2

- z and Laplace transforms, definition, properties
- related recitation problems

Discrete distributions, z transforms

X - discrete r.v., non-negative discrete values $X \in \{0, 1, 2 \dots\}$

PMF: $p_i = P(X=i)$

Generating (or z-) transform: (unilateral z transform, also z^i, not z^{-i})

Def: $G_X(z) = \sum_{i=0}^{\infty} p_i z^i$ (not: $= E[z^X]$)

Why?
- compact way of representing a distribution
- some operations, like convolution ($P(X+Y)$) are simpler
- simplifies the solution of recursive equations.

Properties

a) Linear property $\underbrace{a f_k + b g_k}_{\text{Linear combination}} \Leftrightarrow a F(z) + b G(z)$

b) Convolution property

$H = X + Y$, $\underset{f_X}{X}, \underset{g_Y}{Y}$ iid r.v. $\Leftrightarrow H(z) = F(z) G(z)$

$$H(z) = \sum_{k=0}^{\infty} h_k z^k = \sum_{k=0}^{\infty} \underbrace{\sum_{i=0}^{k} f_i g_{k-i}}_{\text{convolution}} \cdot z^k =$$

$$\sum_{i=0}^{\infty} \sum_{k=i}^{\infty} f_i g_{k-i} z^k = \sum_{i=0}^{\infty} f_i z^i \sum_{k=i}^{\infty} g_{k-i} z^{k-i} = F(z) G(z)$$

c) ~~Expectations~~ Moments

$$G(1) = \sum_{i=0}^{\infty} p_i z^i = \sum_{i=0}^{\infty} p_i = 1$$

$$E[X] = \frac{d}{dz} G(z)\Big|_{z=1} \quad \text{Proof: } \frac{d}{dz} G(z)\Big|_1 = \sum_{i=0}^{\infty} i \cdot p_i z^{i-1}\Big|_{z=1} = \sum_{i=0}^{\infty} i p_i = E[X]$$

~~$E[X^2] = \frac{d^2}{dz^2} G(z)\Big|_{z=1}$ Proof $\frac{d}{dz} G(z)\Big|_1 =$~~

Home!

$$E[x^2] = \frac{d^2}{dz} G(z)\Big|_{z=1} + \frac{d}{dz} G(z)\Big|_{z=1}$$

Proof:

$$= \sum_{i=0}^{\infty} i(i-1) p_i z^{i-2}\Big|_{z=1} + \sum_{i=0} i \cdot p_i z^{i-1}\Big|_{z=1} = \sum_i i^2 p_i - \cancel{\sum_i i p_i} + \cancel{\sum_i i p^i} = E[x^2]$$

~~Var[x] = E[x^2] - E[x]^2~~

(1.3.) Poisson distribution: $p_k = P(x=k) = \frac{a^k}{k!} e^{-a}$ $k = 0, 1, 2 \dots$ $a > 0$

a) Prove that $\sum_{k=0}^{\infty} p_k = 1$ (That is, probability distr.)

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{-a} = e^a \cdot e^{-a} = 1 \checkmark$$

b) $$P(z) = \sum_{k=0}^{\infty} p_k \cdot z^k = \sum_{k=0}^{\infty} \frac{a^k}{k!} e^{-a} \cdot z^k = e^{z \cdot a} \cdot e^{-a} = e^{-a(1-z)}$$ group

c) $$E[x] = \frac{d}{dz} P(z)\Big|_{z=1} = \frac{d}{dz} e^{-a(1-z)}\Big|_{z=1} = a e^{-a(1-z)}\Big|_{z=1} = a$$

$$E[x^2] = \dots a^2 e^{-a(1-z)} + a e^{-a(1-z)} = a^2 + a$$

$$Var[x] = E[x^2] - E[x]^2 = a$$

Translate it to Poisson process:

$$p_k = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad E[x] = \lambda t, \quad E[x^2] = (\lambda t)^2 - \lambda t \quad Var[x] = \lambda t$$

(1.4.) Poisson distribution, sum of random variables

$$X = \sum_1^n X_i, \qquad X_i \sim \text{Poisson}(a_i)$$

$$G_X(z) = \prod_1^n G_{X_i}(z) = \prod_1^n e^{-a_i(1-z)} = e^{-\sum a_i(1-z)} \Leftrightarrow \text{Poisson}\left(\sum_1^n a_i\right)$$

Continuous distributions, Laplace transform

X: non-negative cont. r.v. PDF $f(x)$

Laplace transform: $f^*(s)$, $\mathcal{L}_x(s)$, $F^*(s)$

Def: $f^*(s) = \int_0^\infty e^{-sx} f(x) \quad (= E[e^{-sX}])$

Properties

a) Linear property $\quad a f_x(x) + b g_x(x) \iff a f^*(s) + b g^*(s)$

b) Convolution: $H = X + Y \quad h^*(s) = f^*(s) \cdot g^*(s)$ (independent!)

$X \to f$, $Y \to g$

c) Moments

$E[X] = -\frac{d}{ds} f^*(s)\Big|_{s=0}$

$E[X^2] = \frac{d^2}{ds^2} f^*(s)$

Proof: $-\frac{d}{ds} \int_0^\infty e^{-sx} f(x) dx = -\int_0^\infty -x e^{-sx} f(x) dx$

$= -E[-Xe^{-sX}]\Big|_{s=0} = E[X]$

(1.5) Exponential distribution

$F(x) = P(X \le x) = 1 - e^{-ax}, \quad x \ge 0$

a) $f(x) = \frac{dF(x)}{dx} = \frac{d}{dx}(1 - e^{-ax}) = a \cdot e^{-ax}$ (In class $F(x) = 1 - e^{-\lambda x}$, $f(x) = \lambda e^{-\lambda x}$)

c) $F^*(s) = \int_0^\infty e^{-sx} f(x) dx = \int_0^\infty a \cdot e^{-ax} e^{-sx} dx = a \int_0^\infty e^{-(s+a)x} dx = \frac{a}{s+a}$

d) $E[X]$, $E[X^2]$, $Var[X]$ with Laplace transform

$E[X] = -\frac{d}{ds} \frac{a}{s+a}\Big|_{s=0} = -a(-1)\frac{1}{(s+a)^2}\Big|_{s=0} = \frac{1}{a}$ (in class $E[X] = \frac{1}{\lambda}$)

$E[X^2] = +\frac{d^2}{ds^2} \frac{a}{s+a}\Big|_{s=0} = \frac{a2}{(s+a)^3}\Big|_{s=0} = \frac{2}{a^2}$ (in class $E[X^2] = \frac{2}{\lambda^2}$)

$Var[X] = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$