Recitation 3

Chapter 2:

$\boxed{2.6}$

Chapter 3:

$\boxed{3.3.}$

$\boxed{3.4}$

$\boxed{3.5}$

Poisson process: $P_k(t) = \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t}$

$E[X|t] = \lambda t$

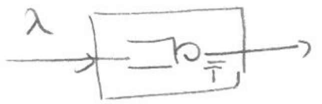
$\tau \sim Exp(\lambda)$

$f(t) = \lambda \cdot e^{-\lambda t}$, $F(t) = 1 - e^{-\lambda t}$, $E[T] = \frac{1}{\lambda}$

Markov chains, global and local balance equations

Littles theorem:

- relation between the arrival rate of customers, average number of customers in the system and the mean time that customers spend in the system.



$\bar{N} = \lambda \cdot \bar{T}$

$N_q = \lambda \cdot W$

$N_s = \lambda \cdot \bar{x}$

$T = \bar{x} + W$

N - avg. num. of custom. in the system

N_q - -||- -||- -||- -||- in the queue

$\bar{x}$ - -||- -||- -||- -||- under service

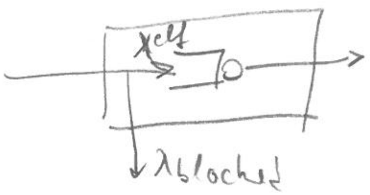
$\bar{T}$ - avg. time spent in the system

W - -||- -||- -||- -||- queue

$\bar{x}$ - avg. service time

- Useful because:
  - Nothing is assumed about the system
  - The arrival process can be anything

Littles theorem for system with blocking



$N = \lambda_{eff} \cdot T$ T: average system time for accepted customers

$N = \lambda \cdot T'$ T' -||- -||- -||- -||- including Ø for blocked customers

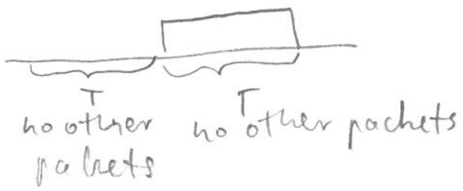
2.6)

- For simplicity we assume packets of constant transmission time T

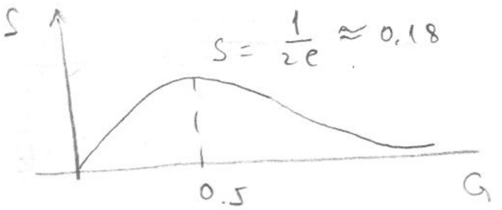
- We assume that packet arrivals + retransmitted packets follow Poisson (g) distribution

g: attempt rate $\left[\frac{\text{packets}}{\text{second}}\right]$ $G = g \cdot T$

Throughput: $S = gT \cdot P(\text{success})$



$$S = gT \cdot P(\text{no packets in } 2T) = gT \cdot e^{-g2T} = G \cdot e^{-2G}$$



3.3)

=4 Holding time: $T \sim Exp(\mu)$ $\qquad f(t) = \begin{cases} \mu \cdot e^{-\mu t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ (1)

$$E[T] = \int_0^\infty t \cdot f(t)\, dt = \int_0^\infty \mu t \cdot e^{-\mu t}\, dt = \frac{1}{\mu}$$

Arrival Process: $\underbrace{P[H_c = k \mid T = t]}$ = P(arriving calls during a conversation of lenght t = k) =

$$= \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t}$$

$$E[H_c \mid t] = \sum_{k=0}^\infty k \cdot P[H_c = k \mid T = t] = \sum_{k=0}^\infty k \cdot \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t} = \lambda t$$

$$P[H_c = k] = \int_0^\infty P[H_c = k \mid T = t] \cdot f(t)\, dt =$$

$$E[H_c] = \sum_{k=0}^\infty k \cdot P[H_c = k] = \sum_{k=0}^\infty k \cdot \int_0^\infty P[H_c = k \mid T = t] f(t)\, dt =$$

(finite)

$$= \int_0^\infty \underbrace{\sum_{k=0}^\infty k \cdot P[H_c = k \mid T = t]}_{E[H_c \mid t] = \lambda t} \cdot f(t)\, dt = \int_0^\infty \lambda t \cdot f(t)\, dt =$$

$$= \lambda \underbrace{\int_0^\infty t \cdot f(t)\, dt}_{E[T] = \frac{1}{\mu}} = \frac{\lambda}{\mu}$$

3.1) $c = 4.8\ k\ \frac{bit}{s}$ ①

$= \kappa$

Arrival: Poisson (λ) $\lambda = 10\ \frac{msg}{s}$

$p_1 = 0.5$ Type 1: $L_1 \sim Exp,\ \overline{L_1} = 300 bits \Rightarrow T_1 \sim Exp(\mu_1)\ E[T_1] = \frac{1}{\mu_1} = \frac{\overline{L_1}}{c} = \frac{1}{16}\ s$

$p_2 = 0.5$ Type 2: $L_2 \sim Exp,\ \overline{L_2} = 150 bits \Rightarrow T_2 \sim Exp(\mu_2)\ E[T_2] = \frac{1}{\mu_2} = \frac{\overline{L_2}}{c} = \frac{1}{32}\ s$

- The buffer can at most hold 1 msg of Type 1 or 2 msgs of Type 2
- A message being transmitted still takes up a place in the buffer.

a.) $E[T] = 0.5 \cdot E[T_1] + 0.5 \cdot E[T_2] = \frac{3}{64}\ s$

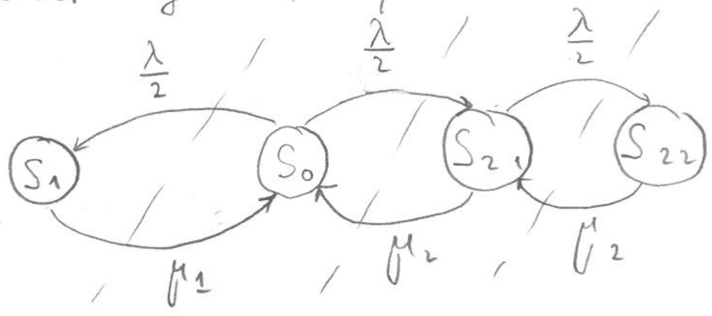
b.) Let us draw a Markov-chain of this system

States: Empty S_0

1 Type 1 S_1

1 Type 2 S_{21}

2 Type 2 S_{22}



Balance equations:

$$p_1 \mu_1 = p_0 \cdot \frac{\lambda}{2} \qquad p_1 \cdot 16 = p_0 \cdot 5$$

$$p_0 \cdot \frac{\lambda}{2} = p_{21} \cdot \mu_2 \qquad p_0 \cdot 5 = p_{21} \cdot 32$$

$$p_{21} \cdot \frac{\lambda}{2} = p_{22} \cdot \mu_2 \qquad p_{21} \cdot 5 = p_{22} \cdot 32$$

$$\Rightarrow \quad p_1 = p_0 \cdot \frac{5}{16}, \quad p_{21} = p_0 \cdot \frac{5}{32}, \quad p_{22} = p_{21} \cdot \frac{5}{32} = p_0 \cdot \frac{25}{(32)^2}$$

$$p_0 + p_1 + p_{21} + p_{22} = 1$$

$$\Rightarrow p_0 \cdot \left(1 + \frac{5}{16} + \frac{5}{32} + \frac{25}{(32)^2}\right) = 1$$

$$p_0 \left(\frac{32^2 + 5 \cdot 2 \cdot 32 + 5 \cdot 32 + 25}{32^2}\right) = 1$$

$$p_0 = \frac{1024}{1529} \approx 0.670$$

$$p_1 \approx 0.209$$

$$p_{21} \approx 0.105$$

$$p_{22} \approx 0.016$$

$E[T_1 | \text{accepted}] = E[\text{transmission time}] = \frac{1}{\mu_1} \approx 62{,}5 \text{ ms}$ (2)

state S_0 only

prob that msg will arrive to S_0 if accepted

$E[T_2 | \text{accepted}] = E[T_2 | S_0] \cdot P[S_0 | \text{accepted}] +$

states S_0 and S_{21}

$+ E[T_2 | S_1] \cdot P[S_1 | \text{accepted}] =$

prob that msg will arrive to S_1 if accepted

$= E[\text{transmission time}] \cdot \frac{P_0}{P_0 + P_{21}} +$

$+ E[\text{waiting time} + \text{transmission time}] \cdot \frac{P_{21}}{P_0 + P_{21}} =$

$= \frac{1}{16} \cdot \frac{P_0}{P_0 + P_{21}} + \frac{2}{16} \cdot \frac{P_{21}}{P_0 + P_{21}} \approx 35{,}5 \text{ ms}$

c.) P(message arrive and it is dropped)

$P(\text{dropped} | \text{Type } 1) = 1 - P_0 \approx 0.33$

it is not dropped only in state S_0

$P(\text{dropped} | \text{Type } 2) = P_1 + P_{22} \approx 0.225$

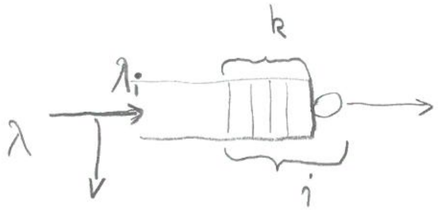
3.5) $\lambda = \frac{1}{7} \frac{\text{jobs}}{s} \to$ Poisson (1.)

$\mu = \frac{1}{6} \frac{\text{jobs}}{s} \to$ Exp.

single server, State: # jobs in the system

P(do not join the queue | k) = l_k k - number of jobs in the queue

$$l_k = \begin{cases} \frac{k}{4} & k < 4 \\ 1 & k \geq 4 \end{cases} \qquad l_0 = 0,\ l_1 = \frac{1}{4},\ l_2 = \frac{2}{4},\ l_3 = \frac{3}{4},\ l_4 = l_5 = \dots = 1$$

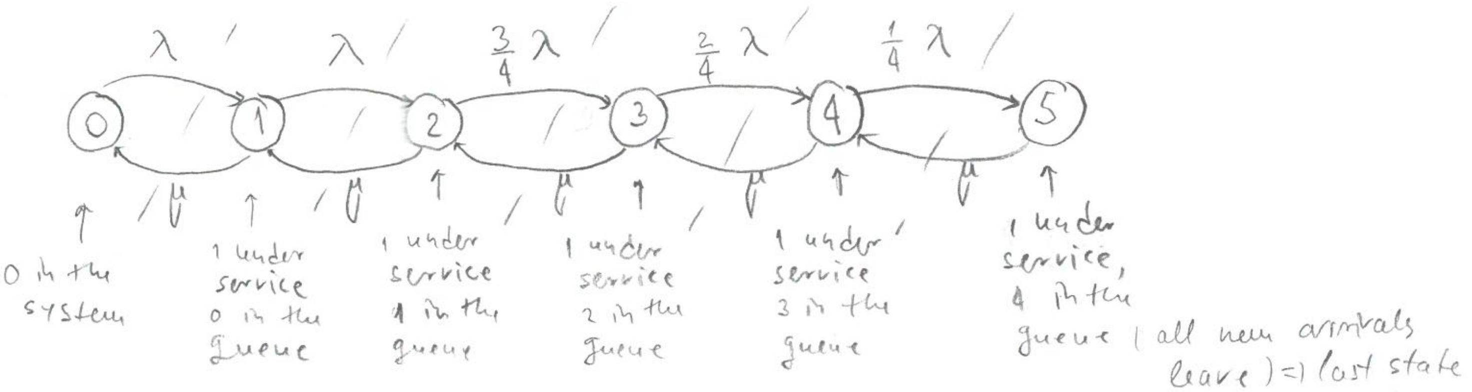


$\lambda_i = (1 - l_k) \cdot \lambda$

a.) $\bar{N}$ - mean number of customers in the system

b.) # of jobs served in 100 s (J_{100})

=) We need to know the steady state probabilities:



Balance equations

$$\left.\begin{aligned} p_0 \cdot \lambda &= p_1 \cdot \mu \\ p_1 \lambda &= p_2 \cdot \mu \\ p_2 \cdot \tfrac{3}{4} \lambda &= p_3 \mu \\ p_3 \cdot \tfrac{1}{2} \lambda &= p_4 \cdot \mu \\ p_4 \cdot \tfrac{1}{4} \lambda &= p_5 \cdot \mu \\ p_0 + p_1 + p_2 + p_3 + p_4 + p_5 &= 1 \end{aligned}\right\} \quad p_0 \approx 0.3$$

a =) $N = \sum_{i=0}^{5} i \cdot p_i \approx 1.43$

b.) In state S_0 the server is idle (=)

When not in S_0 it serves the jobs

$$J_{100} = \overbrace{(1 - p_0)}^{\Downarrow} \cdot \underbrace{\frac{100\,s}{\bar{X}}}_{\uparrow} = (1 - p_0) \cdot \mu \cdot 100 \approx 11.66$$

the number of jobs if the server is busy in each state