Recitation 2

Chapter 1:

$\boxed{1.3}$ a.) c.) ($E[X]$, $E[X^2]$, $Var[X]$) without z-transform

$\boxed{1.5}$ a.) d.) ($E[X]$, $E[X^2]$, $Var[X]$) without Laplace-transform

$\boxed{1.6}$ a.) b.)

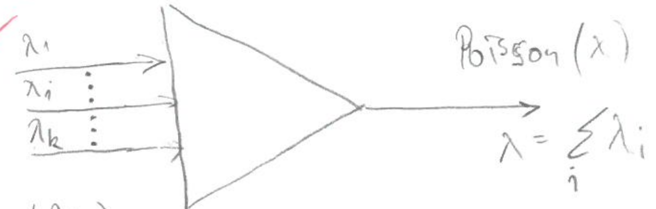
Chapter 2:

$\boxed{2.2}$

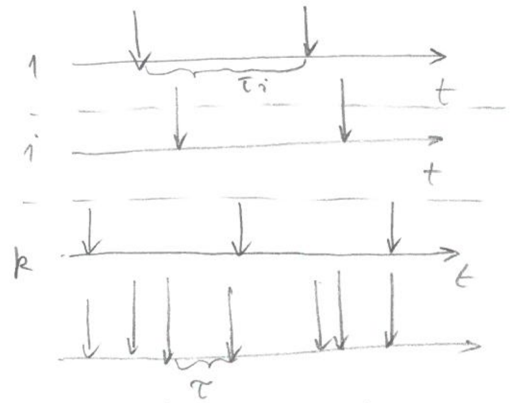
$\boxed{2.3}$

$\boxed{2.4}$

2,2)



Poisson (λ_i)



- Combined packet stream is still Poissonian with $\lambda = \sum_{i=1}^{k} \lambda_i$

- τ : interarrival time for the combined packet stream

$P(\tau > t) = P($ no arrival in any of the inputs in time period $t) \underset{\substack{\uparrow \\ \text{processes} \\ \text{are independant}}}{=}$

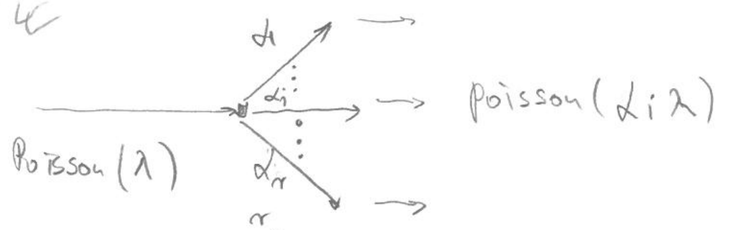
$\prod_{i=1}^{k} P($ no arrival at input i in time period $t) =$

$= \prod_i \underbrace{e^{-\lambda_i \cdot t}}_{\text{CCDF of } Exp(\lambda_i)} = \underbrace{e^{-\sum_{i=1}^{k} \lambda_i \cdot t}}_{\text{CCDF of } Exp(\sum_{i=1}^{k} \lambda_i)}$

$\Rightarrow$ Intraarrival time for the combined packet stream follows $Exp \sim (\sum_{i=1}^{k} \lambda_i)$ distribution $\Rightarrow$ Combined packet stream is still Poissonian with

$\lambda = \sum_{i=1}^{k} \lambda_i$.

2.3 4



$$\sum_{i=1}^{r} \alpha_i = 1$$

- When an event happens, we assign it randomly to one of the outgoing link

τ_i - interarrival time for the stream i

$P(\tau_i > t) = P($no arrivals in the main stream in time period $t) + \sum_{k=1}^{\infty} P($k arrivals, but sent to the other outputs$) =$

Poisson process

$$e^{-\lambda t} + \sum_{k=1}^{\infty} (1-p_i)^k \cdot \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t} = \sum_{k=0}^{\infty} \frac{[(1-p_i)\cdot\lambda t]^k}{k!} \cdot e^{-\lambda t}$$

$$\left[\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a \right]$$

$$P(\tau_i > t) = e^{(1-p_i)\lambda t} \cdot e^{-\lambda t} = \underbrace{e^{-p_i \cdot \lambda t}}_{\text{CCDF of } Exp(p_i \cdot \lambda)}$$

- Interarrival time for the stream i follows $Exp \sim (p_i \cdot \lambda)$ distribution $\Rightarrow$ Outgoing packet stream is Poisson with $\lambda_i = p_i \cdot \lambda$.

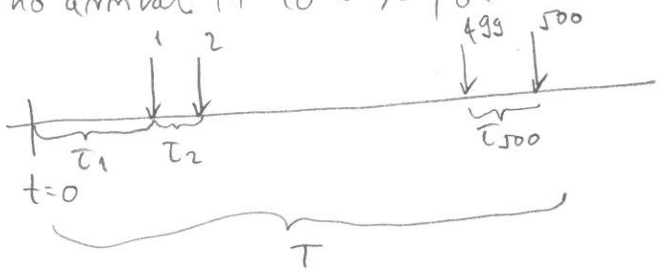
2.4) Poisson arrival process $\lambda = 1000 \frac{\text{pack.}}{s}$ (1)

$P_k(t) = \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t}$, interarrival $\tau \sim Exp(\lambda)$

$E[\tau] = \frac{1}{\lambda}$ $Var[\tau] = \frac{1}{\lambda^2}$

a.) $P(\text{no arrival in } 10^{-3}s) = P_0(10^{-3}) = e^{-10^3 \cdot 10^{-3}} = e^{-1} \approx 0.36$

b.)



$E[T] = ?$ $Var[T] = ?$

$T = \sum_{i=1}^{500} \tau_i$ $\Rightarrow$ (Sum of n independent r.v. with $Exp(\lambda)$ distr. is Erlang(n, λ) distribution.)

$E[\tau] = \frac{1}{\lambda}$ $(= 1\,ms)$

$E[T] = \sum_{i=1}^{500} E[\tau] = 500\,ms$

$Var[T] = \sum_{i=1}^{500} Var[\tau] = 500 \cdot 10^{-6}\,s^2$

1.3 $X \sim$ discrete stochastic variable

$$p_k = P(X=k) = \frac{a^k}{k!}\cdot e^{-a}, \quad k = 0, 1, 2, \ldots$$

$\underline{a>0}$

a.) Prove that $\sum_{k=0}^{\infty} p_k = 1$

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{a^k}{k!}\cdot e^{-a} = e^{a}\cdot e^{-a} = 1$$

$$\boxed{\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^{a}} \rightarrow \text{exponential series formula}$$

b.)

$$\underbrace{P(z) = \sum_{k=0}^{\infty} p_k \cdot z^k}_{\uparrow} = \sum_{k=0}^{\infty} \frac{a^k}{k!}\cdot e^{-a}\cdot z^k = e^{za}\cdot e^{-a} = \underline{e^{-a\cdot(1-z)}}$$

z-transform def

c.) Without using z-transform:

$$* \; E[X] = \sum_{k=0}^{\infty} k\cdot p_k = \sum_{k=1}^{\infty} k\cdot\frac{a^k}{k!}\cdot e^{-a} = \sum_{k=1}^{\infty} \frac{a^k}{(k-1)!}\cdot e^{-a} = a\cdot e^{-a}\sum_{k=1}^{\infty}\frac{a^{k-1}}{(k-1)!} =$$

↑ for k=0 there is no contribution

$$= a\cdot e^{-a}\sum_{j=0}^{\infty}\frac{a^j}{j!} = a\cdot e^{-a}\cdot e^{a} = \underline{\underline{a}}$$

↑ $j = k-1$

$$* \; E[X^2] = \sum_{k=0}^{\infty} k^2\cdot p_k = \sum_{k=1}^{\infty} \cancel{k}^2\cdot\frac{a^k}{k!}\cdot e^{-a} = \sum_{k=1}^{\infty}(k-1)\cdot\frac{a^k}{(k-1)!}\cdot e^{-a} + \underbrace{\sum_{k=1}^{\infty}\frac{a^k}{(k-1)!}\cdot e^{-a}}_{a} =$$

↑ $(\underbrace{k-1+1 = k})$

$$= \sum_{k=2}^{\infty}(k-1)\cdot\frac{a^k}{(k-1)!}\cdot e^{-a} + a = a^2\cdot e^{-a}\sum_{k=2}^{\infty}\frac{a^{k-2}}{(k-2)!} + a =$$

↑ $j = k-2$

for k=0 there is no contribution

$$= a^2\cdot e^{-a}\sum_{j=0}^{\infty}\frac{a^j}{j!} + a = a^2\cdot e^{-a}\cdot e^{a} + a = \underline{a^2+a}$$

(2)

$Var[X] = E[X^2] - E[X]^2 = \underline{a}$

* $E[X\cdot(X-1)\dots(X-r+1)] = \sum_{k=0}^{\infty} k\cdot(k-1)\dots(k-r+1)\cdot p_k =$

$$= \sum_{k=0}^{\infty} k\cdot(k-1)\dots(k-r+1)\cdot \frac{a^k}{k!}\cdot e^{-a} =$$

$$= \sum_{k=r}^{\infty} \cancel{k\cdot(k-1)\dots(k-r+1)}\cdot \frac{a^k}{k!}\cdot e^{-a} =$$

$$= a^r\cdot e^{-a} \sum_{k=r}^{\infty} \frac{a^{k-r}}{(k-r)!} \underset{j=k-r}{=}$$

$$= a^r\cdot e^{-a}\cdot \sum_{j=0}^{\infty} \frac{a^j}{j!} = a^r$$

- With z-transform:

* $E[X] = \frac{d}{dz} P(z)\Big|_{z=1} = \frac{d}{dz} e^{-a\cdot(1-z)}\Big|_{z=1} = a\cdot e^{-a\cdot(1-z)}\Big|_{z=1} = \underline{a}$

* $E[X^2] = \frac{d^2}{dz^2} P(z)\Big|_{z=1} + \frac{d}{dz} P(z)\Big|_{z=1} = a^2\cdot e^{-a\cdot(1-z)}\Big|_{z=1} + a\cdot e^{-a\cdot(1-z)}\Big|_{z=1} =$

$$= \underline{\underline{a^2 + a}}$$

!! Translate it to Poisson process:

* $p_k = \frac{(\lambda t)^k}{k!}\cdot e^{-\lambda t}$, $E[X] = \lambda\cdot t$, $E[X^2] = (\lambda t)^2 - \lambda t$, $Var[X] = \lambda t$

1.5 $X \sim Exp(a)$, $a > 0$ ①

$$F(x) = P(X \le x) = \begin{cases} 0 & , \quad x < 0 \\ 1 - e^{-ax} & , \quad x \geq 0 \end{cases}$$

a.) pdf $f(x) = \dfrac{d\,F(x)}{dx} = \dfrac{d}{dx}\left(1 - e^{-ax}\right) = a \cdot e^{-ax}$

b.) $\bar{F}(x) = P(X > x) = 1 - P(X \le x) = \begin{cases} 1 & , \quad x < 0 \\ e^{-ax} & , \quad x \geq 0 \end{cases}$

c.) Laplace transform

$$F^*(s) = \int_0^\infty e^{-sx} \cdot f(x) \cdot dx = \int_0^\infty a \cdot e^{-ax} \cdot e^{-sx}\, dx = a \int_0^\infty e^{-(s+a)\cdot x}\, dx = \frac{a}{s+a}$$

d.) Without Laplace transform:

$$E[X] = \int_0^\infty x f(x) \cdot dx = \int_0^\infty x \cdot a \cdot e^{-ax}\, dx = a \cdot e^{-ax} \cdot \left(\frac{-ax - 1}{(-a)^2}\right)\Bigg|_0^\infty =$$

$$= a \cdot e^{-ax} \cdot \left(\frac{-ax - 1}{a^2}\right)\Bigg|_0^\infty = 0 - \left(-\frac{1}{a}\right) = \frac{1}{a}$$

$$E[X^2] = \int_0^\infty x^2 \cdot f(x) \cdot dx = \int_0^\infty x^2 \cdot a \cdot e^{-ax}\, dx = a \cdot e^{-ax} \cdot \left(\frac{x^2}{-a} - \frac{2x}{(-a)^2} + \frac{2}{(-a)^3}\right)\Bigg|_0^\infty =$$

$$= 0 - a \cdot \frac{2}{-a^3} = \frac{2}{a^2}$$

$$Var[X] = E[X^2] - E[X]^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$

$$\left[\begin{aligned} &\int e^{cx}\, dx = \frac{1}{c} \cdot e^{cx} \\ &\int x \cdot e^{cx}\, dx = e^{cx} \cdot \left(\frac{cx - 1}{c^2}\right) \\ &\int x^2 \cdot e^{cx}\, dx = e^{cx} \cdot \left(\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}\right) \end{aligned}\right]$$

With Laplace transform: (2.)

$$E[X] = -\frac{d}{ds} F^*(s)\Big|_{s=0} = -\frac{d}{ds}\cdot\frac{a}{s+a}\Big|_{s=0} = -\frac{0\cdot(s+a)-1\cdot a}{(s+a)^2}\Big|_{s=0} = \frac{a}{(s+a)^2}\Big|_{s=0} = \frac{1}{a}$$

$$E[X^2] = +\frac{d^2}{ds^2} F^*(s)\Big|_{s=0} = \frac{0\cdot(s+a)^2 - 2\cdot(s+a)\cdot(-a)}{(s+a)^4}\Big|_{s=0} = \frac{2a}{(s+a)^3}\Big|_{s=0} = \frac{2}{a^2}$$

!! Translate it to interarrival times:

$E[\tau] = \frac{1}{\lambda}$ $E[\tau^2] = \frac{2}{\lambda^2}$ $Var[\tau] = \frac{1}{\lambda^2}$

ok

1.6 X_i - i.d.i r.v $E[X_i] = \frac{1}{a}$ for $\forall i = 1, \dots n)$, $a > 0$ (1.)

Calculate $P(X \le x)$ and $P(X \ge x)$, $x \ge 0$

a.) $X = \min(X_1, X_2, \dots, X_n)$

$X_i \sim Exp(a)$

$$\bar{F}(x) = P(X > x) \underset{\substack{\uparrow \\ \text{if min} > x\text{, then} \\ \text{all} > x}}{=} P(\text{all } X_i > x) \overset{\substack{\text{i.d.i.} \\ \downarrow}}{=} \prod_{i=1}^{n} P(X_i > x) = \prod_{i=1}^{n} e^{-ax} =$$

$$= e^{-\sum_{i=1}^{n} ax} = e^{-nax}, \quad x \ge 0$$

$$F(x) = P(X \le x) = 1 - P(X > x) = 1 - e^{-nax}, \quad x \ge 0$$

$\Rightarrow X \sim Exp(na)$

- More general: if $X_i \sim Exp(a_i)$

$\Rightarrow X \sim Exp\left(\sum_i a_i\right)$

b.) $X = \max(X_1, X_2, \dots, X_n)$

$$F(x) = P(X \le x) \underset{\substack{\uparrow \\ \text{if max} \le x\text{, then} \\ \text{all} > x}}{=} P(\text{all } X_i \le x) \underset{\substack{\uparrow \\ \text{i.d.i}}}{=} \prod_{i=1}^{n} P(X_i \le x) =$$

$$= \prod_{i=1}^{n} (1 - e^{-ax}) = (1 - e^{-ax})^n, \quad x \ge 0$$

$$\bar{F}(x) = P(X > x) = 1 - P(X \le x) = 1 - (1 - e^{-ax})^n, \quad x \ge 0$$