

Algorithms and Complexity

What is the course about?

How to solve problems in an algorithmic way.

Three examples:

1. Sorting of numbers

← Can be solved easily. But it can be solved more efficiently.

2. Shortest paths in graphs

← A bit tricky to solve efficiently.

3. Partitioning of a set of numbers.

← Can not be solved efficiently.

There is a set of natural questions concerning algorithms:

How do we measure efficiency?

How do we find algorithms?

How do we describe algorithms?

How do we know if an algorithm works?

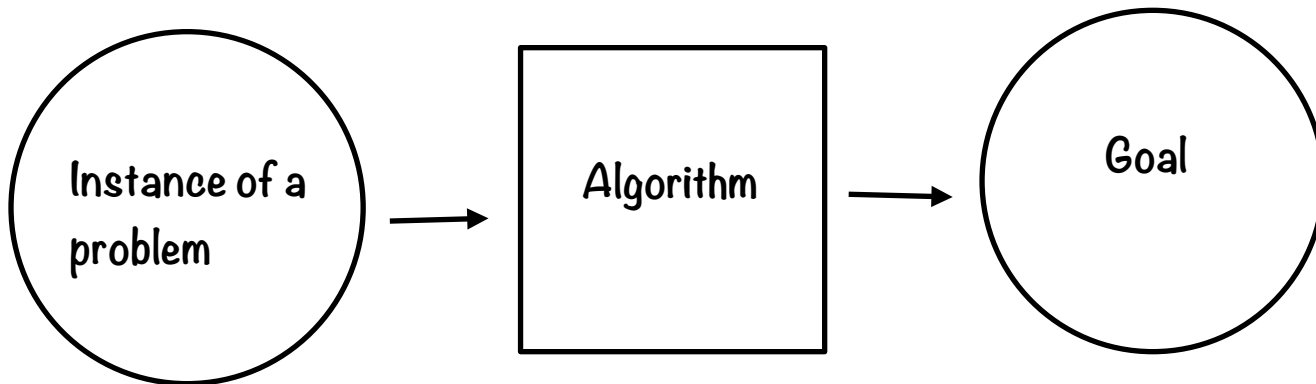
How do we know if a problem can be solved efficiently?

How do we know if a problem can be solved algorithmically at all?

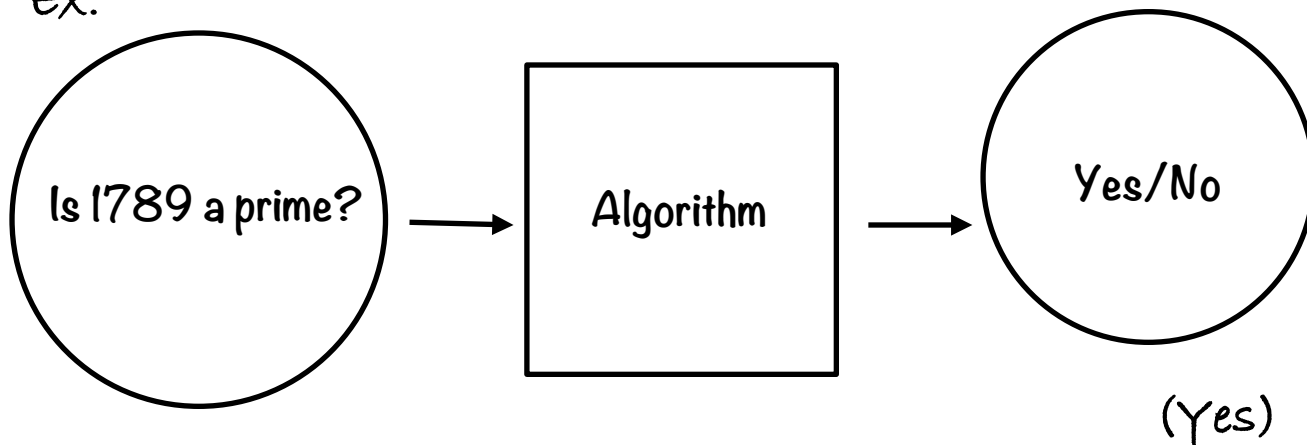
By the way, what is a problem?

We will try to provide some answers.

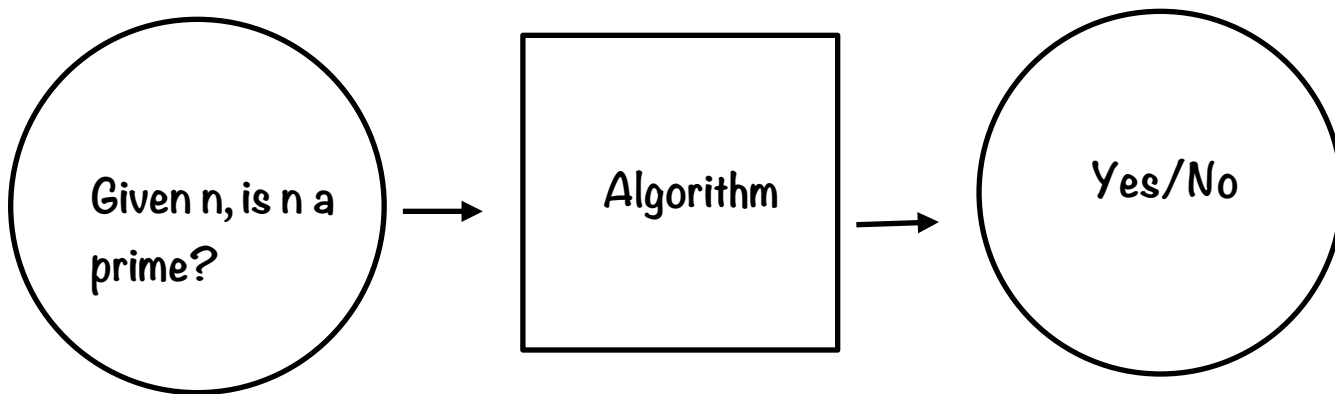
A general picture:



Ex.



But an algorithm is usually not designed to solve just one instance.
Normally it solves an infinite numbers of instances:



So what is a problem?

A problem consist of Input and Goal

Ex:

PRIME NUMBER

Input: An integer n

Goal: Is n a prime number?

SORTING

Input: A list $L = \{l_1, l_2, \dots\}$ of numbers

Goal: A sorting $L' = \{l'_1, l'_2, \dots\}$ such that l'_1, l'_2, \dots is increasing

SHORTEST PATH

Input: A graph G . Two nodes a, b

Goal: A path from a to b with as few edges as possible

How do we describe an algorithm?

An answer would be to just give a program code.

But that can be inconvenient.

Take for instance the most famous sorting algorithm:

Insertion-Sort.

Informal description:

We start with a list $A[1], A[2], \dots$

Take $A[1]$ and set $L_1 = A[1]$.

Take $A[2]$ and sort into L_1 . This gives us a new list L_2 .

Take $A[3]$ and sort into L_2 . This gives us a new list L_3 .

And so on ...

We describe the algorithm with so called pseudocode.

INSERTION-SORT(A)

```
1  for  $j = 2$  to  $A.length$ 
2     $key = A[j]$ 
3    // Insert  $A[j]$  into the sorted sequence  $A[1..j-1]$ .
4     $i = j - 1$ 
5    while  $i > 0$  and  $A[i] > key$ 
6       $A[i + 1] = A[i]$ 
7       $i = i - 1$ 
8     $A[i + 1] = key$ 
```

How do we know that Insertion-Sort works correctly?

Think like this: At step k we know that

$A[1], A[2], \dots, A[k]$ is a part of the start list and
 $A[1], A[2], \dots, A[k]$ is sorted.

When the algorithm stops $A[1], A[2], \dots, A[n]$ must be the sorted list.

Usually, the problem is to decide if a loop works correctly. Sometimes we can define a loop-invariant I that is always true for each step in the loop.

Suppose the loop has the form

While B do
 (...) ← ..
 Something happens here
 .

When the loop ends we have I and not B . If we have chosen I wisely, we will have reached our goal.

What about efficiency? How do we measure it?

Basic idea:

We measure the number of steps needed as a function of the size of the input. This type of measure is called time-complexity.

Insertion-Sort: If we have n elements we need $O(n^2)$ steps.

Problem: How do we define size of input?

Basic idea: We use a string to represent the input. We measure the size of the string.

In practice, we use something that is a convenient approximation of the size of the string.

Ex: If the input is n elements in a list we normally use n as a measure of the input size.

Ex: If the input is just an integer n we use $\log_2 n$ as the size of the input.

But there is a problem with finding good representations.

A bit more about complexity

Time-complexity is not the only measure of complexity.

We can measure the size of the memory space the algorithm needs to complete the computation. This measure is called space-complexity.

Time-complexity more important but space-complexity is also important.

Later we will show that
high space-complexity implies high time-complexity

The opposite is not necessarily true.

Unit cost and bit cost

An interesting question is what counts as a step in a computation.

Unit cost: Each "algebraic operation" counts as one step

Bit cost: Each bit operation counts as a step

Ex:

We want to compute $521 * 394$. What is the cost?

Unit cost: Just one operation! The cost is 1.

Bit cost: 521 has 9 bits. 394 has 8 bits. The cost is $9 * 8 = 72$

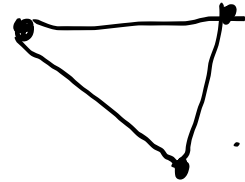
(The answer is 205274.)

Unit cost is easier to work with and we normally use it.

What about efficiency? How do we measure it?
We will study a special problem in graph theory

Input: A graph G and a node s .

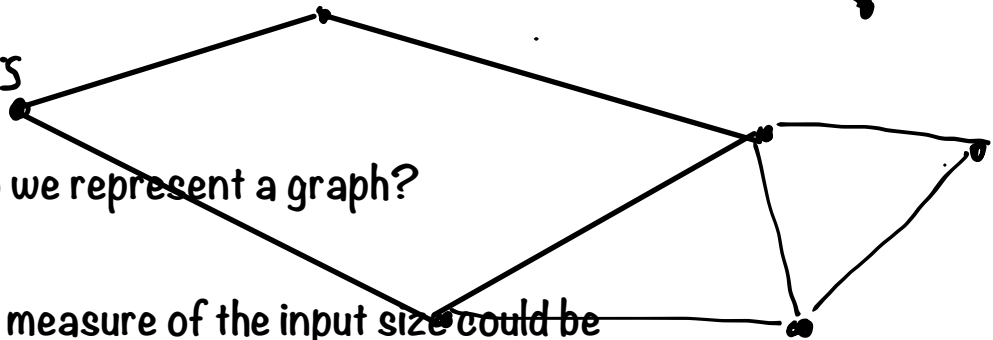
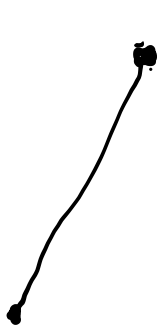
Basic idea:
Goal: Which nodes can be reached by paths from s ?
We measure the number of steps needed as a function of the size of the input.



Insertion-Sort: If we have n elements we need $O(n^2)$ steps

Ex: How do we represent a graph?

The natural measure of the input size could be $|V|$ or $|E|$ or $|V| + |E|$



A simple algorithm
Usually, the complexity is given as a function of $|V|$ and $|E|$.

$R \leftarrow \{s\}$

While there is (u,v) such that $u \in R$ and $v \notin R$

 Add v to R

End while

When the algorithm stops, R is the set of nodes reachable from s .

But there is an uncertainty. How shall (u,v) be chosen?

Data structures for graphs

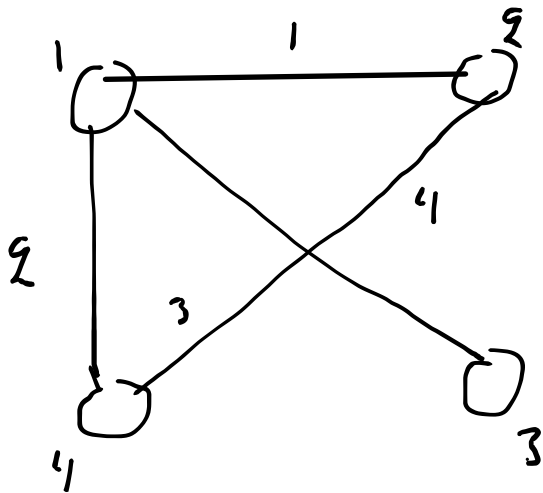
There are basically 3 ways to describe a graph in a form suitable for computation.

Adjacency matrix

Incidence matrix

Adjacency lists

Ex:



adjacency matrix

$A(i,j) = 0/1$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$A(i,j)=1 \iff$ There is an edge (i,j)

Incidence matrix

$$I = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$I(i,j) = 0/1$

$I(i,j)=1 \iff$ nod i is on edge j

Adjacency lists

$j \text{ is in } L_i \iff \text{There is an edge } (i,j)$

$L_1: 2, 3, 4$

$L_2: 1, 4$

$L_3: 1$

$L_4: 1, 2$

We now present an algorithm which is a more detailed solution to our graph problem.

When the algorithm stops, T is a tree containing all nodes that can be reached from s . The algorithm is known as Breadth-First-Search (BFS)

The complexity is $O(V + E)$

```
For all  $v \in V$ 
    set  $\text{vis}(v) = 0$ 
End for
Set  $\text{vis}(s) = 1$ 
Set  $\text{count} = 0$ 
Set  $T = \emptyset$ 
Set  $L[0] = \{s\}$ 
While  $L[i]$  is not empty
    Set  $L[i+1] = \emptyset$ 
    For each  $u \in L[i]$ 
        For each edge  $(u,v)$ 
            If  $\text{vis}(v) = 0$ 
                Set  $\text{vis}(v) = 1$ 
                Add  $(u,v)$  to  $T$ 
                Add  $v$  to  $L[i+1]$ 
            End if
        End for
    End for
    Set  $i = i+1$ 
End while
```

We describe one more search algorithm for graphs: Depth-First Search (DFS)

```
Set =  $\emptyset$ 
For all  $v \in V$ 
    Set  $\text{vis}(v) = 0$ 
End for
DFS(s)
```

```
DFS(u):
    Set  $\text{vis}(u) = 1$ 
    Add u to R
    For each v such that v is adjacent to u
        If  $\text{vis}(v) = 0$ 
            DFS(v)
        End if
    End for
```

The complexity is $O(|V| + |E|)$.
The algorithm is defined recursively. A non-recursive definition is:

```
Set R =  $\emptyset$ 
Let S' be a stack and set S = {s}
For all  $v \in V$ 
    Set  $\text{vis}(v) = 0$ 
End for
Set  $\text{vis}(s) = 1$ 
While S  $\neq \emptyset$ 
    Take the top node u from S
    If  $\text{vis}(s) = 0$ 
        Add u to R
        For each v adjacent to u
            Add v to the top of S
        End for
    End if
End while
```

Efficient algorithms

What is an efficient algorithm? We can start by asking what an inefficient algorithm is. If the size of the input is n and the algorithm works in time $O(2^n)$ we say that the algorithm has exponential complexity. Such an algorithm is clearly very inefficient.

In several situations where we have a problem with an input of size n , we have a set of possible solutions of size exponential in n . Just one (or a few) of these possible solutions are real solutions. If we just test all possible solutions to see if any of them are real solutions we get an algorithm with exponential complexity. In order to get a better algorithm we must find a way of "zooming in" on the real solutions.

It seems natural to say that an efficient algorithm is an algorithm which is not exponential. But we don't do this. Instead we use the following definition:

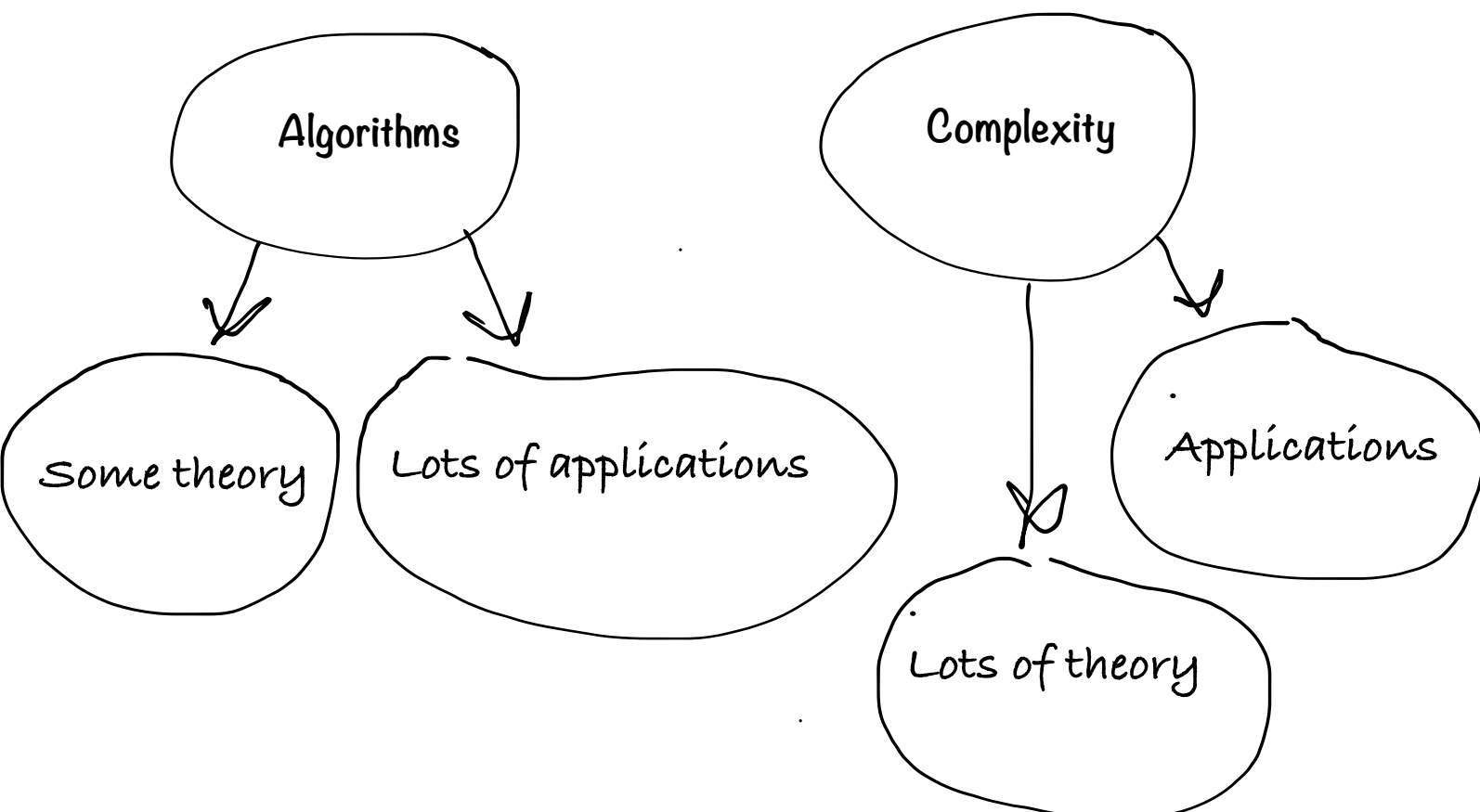
We say that an algorithm has polynomial time-complexity if there is an integer k such that the algorithm, if started with input of size n , runs in $O(n^k)$ steps.

Efficient algorithms:

The standard definition is to say that an algorithm is efficient if and only if it has polynomial time-complexity.

An outline of the course

We have two parts:



Algorithms

We describe several types of algorithms and ways to construct algorithms.

Complexity

Not all problems can be solved efficiently.

Some problems can not be solved at all!

Some of the things we will do in this part are:

Study Turing Machines and formally define computing

Study uncomputable problems.

Study NP-problems which (probably) can not be solved efficiently.

Study so called approximation algorithms.