Department of Mathematics
SF1626 Calculus in Several Variable

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Solutions to the exam 2017-01-10

DEL A

1. A particle moves in such a way that its position is described by
$$(x, y, z) = (t\cos t, t\sin t, t),$$
where the axes measure units in meter and where t is the time measured in seconds, starting from 0.
   (a) Determine the velocity of the particle after 1 second. **(1 p)**
   (b) What is the speed of the particle have after 1 second? **(1 p)**
   (c) What is the arc length of the curve that the particle describes during the first second? **(2 p)**

Solution.

(a) The velocity is the derivative of the position vector with respect to time, i.e.
$$(x'(t), y'(t), z'(t)) = (\cos t - t\sin t, \sin t + t\cos t, 1) \quad \text{m/s}.$$
At time $t = 1$ we have
$$(x'(1), y'(1), z'(1)) = (\cos 1 - \sin 1, \sin 1 + \cos 1, 1) \quad \text{m/s}.$$

(b) Speed is defined as the length of the velocity vector. Thus, the speed after 1s equals
$$\begin{aligned}|(x'(1), y'(1), z'(1))| &= |(\cos 1 - \sin 1, \sin 1 + \cos 1, 1)| \\ &= \sqrt{(\cos 1 - \sin 1)^2 + (\sin 1 + \cos 1)^2 + 1} \\ &= \sqrt{3} \quad \text{m/s}\end{aligned}$$

(c) The arc length of the curve described by the particle is given by
$$\begin{aligned}\int_0^1 |(x'(t), y'(t), z'(t))|\,dt &= \int_0^1 \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}\,dt \\ &= \int_0^1 \sqrt{2 + t^2} \\ &= \frac{\sqrt{3}}{2} + \ln(1 + \sqrt{3}) - \ln\sqrt{2} \quad \text{m.}\end{aligned}$$

Answer.

(a) $(\cos 1 - \sin 1, \sin 1 + \cos 1, 1)$ m/s

(b) $\sqrt{3}$ m/s

(c) $\frac{\sqrt{3}}{2} + \ln(1+\sqrt{3}) - \ln\sqrt{2}$ m.

2. Determine whether or not the vector field $\mathbf{F}(x, y) = (\sin x \sin y, -\cos x \cos y)$ is conservative and then compute the integral
$$\int_C \sin x \sin y \, dx - \cos x \cos y \, dy$$
along the curve C from $(0, 0)$ to $(1, 1)$ that is given by $y = x^2$. **(4 p)**

Solution. In order to determine whether the vector field is conservative, we need to find out whether $\mathbf{F}$ has a potential ϕ. The conditions that such a potential ϕ has to satisfy are
$$\frac{\partial \phi}{\partial x} = \sin x \sin y \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = -\cos x \cos y.$$
We observe that the function
$$\phi(x, y) = -\cos x \sin y$$
satisfies both conditions everywhere on the plane. Thus, there exists a potential and the vector field is conservative.

We may compute the line integral with the help of the potential that we found:
$$\int_C \sin x \sin y \, dx - \cos x \cos y \, dy = \phi(1, 1) - \phi(0, 0) = -\cos 1 \sin 1.$$

Answer. The vector field is conservative and the line integral equals $-\cos 1 \sin 1$.

3. Consider the function $f(x, y) = 2x^3 - 6xy^2 + y^4$.
   (a) Show that $(0, 0)$, $(3, -3)$ and $(3, 3)$ are the only critical points of f. **(1 p)**
   (b) Find the Taylor polynomial of degree two of f at the critical point $(3, -3)$. Use them to decide whether the points are local minima, local maxima or saddle points. **(2 p)**
   (c) Explain why the Taylor polynomial of degree two at the origin cannot be used to determine whether the origin is a local minimum or maximum or a saddle point. **(1 p)**

Solution.

(a) Since f is a polynomial, it can be differentiated arbitrarily often. We have

$$\frac{\partial f}{\partial x} = 6x^2 - 6y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = -12xy + 4y^3.$$

The critical points arise as the solutions to $\partial f/\partial x = \partial f \partial y = 0$. The first equation, $\partial f/\partial x = 0$, yields $y = \pm x$. Inserting this into the second equation, we obtain $12x^2 = 4x^3$, which has the solutions $x = 0$ and $x = 3$. Since $y = \pm x$, the only critical points are thus $(0, 0)$, $(3, 3)$ and $(3, -3)$.

(b) In order to determine the relevant Taylor polynomial, we need to compute all second-order derivatives of f. These are

$$\frac{\partial^2 f}{\partial x^2} = 12x, \quad \frac{\partial^2 f}{\partial x \partial y} = -12y, \quad \frac{\partial^2 f}{\partial y^2} = -12x + 12y^2.$$

Combining the above with the information from part (a), the Taylor polynomial of degree two about $(3, -3)$ is seen to equal

$$p_{3,-3}(x, y) = -27 + \frac{1}{2}\left(36(x-3)^2 + 2 \cdot 36(x-3)(y+3) + 72(y+3)^2\right).$$

Setting $x = 3 + h$ and $y = -3 + k$, this becomes

$$p_{3,-3}(3 + h, -3 + k) = -27 + \frac{1}{2}\left(36h^2 + 2 \cdot 36hk + 72k^2\right).$$

Completing the quadratic form to a square leads to

$$\frac{1}{2}\left(36h^2 + 2 \cdot 36hk + 72k^2\right) = 18((h + k)^2 + k^2),$$

which is positive definite. This shows that the critical point $(3, -3)$ is a local minimum.

(c) The Taylor polynomial of degree 2 about the origin is given by $p_{0,0} = 0$. In this case, the quadratic form is semi-definite and does not provide any information (all second-order derivatives of f at the origin are 0).

Answer.

(a) See the solution above.
(b) $(3, -3)$ is a local minimum.
(c) The Taylor polynomial is zero so the quadratic form is semi-definite.

Del B

4. Let D be the domain in the plane that is described by the inequalities

$$x^2 + y^2 \leq 1 \quad \text{and} \quad x \leq |y|.$$

Determine the centre of gravity of the domain D. **(4 p)**

Solution. In polar coordinates, D is described by

$$\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}, \qquad 0 \leq r \leq 1.$$

For reasons of symmetry, the y-coordinate of the centre of gravity must be zero. Since D consists of three quarters of the unit disk, its area is given by $\frac{3\pi}{4}$. The x-coordinate of the centre of gravity is therefore given by

$$\begin{aligned}
\frac{\iint_D x\,dA}{\iint_D dA} &= \frac{4}{3\pi} \int_{\pi/4}^{7\pi/4} \left(\int_0^1 r^2 \cos\theta \, dr \right) d\theta \\
&= \frac{4}{3\pi} \cdot \left(\int_{\pi/4}^{7\pi/4} \cos\theta \, d\theta \right) \cdot \left(\int_0^1 r^2 \, dr \right) \\
&= \frac{4}{3\pi} \cdot \left(\sin\frac{7\pi}{4} - \sin\frac{\pi}{4} \right) \cdot \frac{1}{3} \\
&= \frac{4}{3\pi} \cdot \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) \cdot \frac{1}{3} \\
&= -\frac{8}{9\pi\sqrt{2}}.
\end{aligned}$$

Thus, the centre of gravity of D is the point $(-8/9\pi\sqrt{2}, 0)$.

Answer. The centre of gravity of D is the point $(-8/9\pi\sqrt{2}, 0)$.

5. *The hyperbolic coordinates* of the plane are defined by

$$(u,v) = \phi(x,y) = \left(\ln\sqrt{\frac{x}{y}}, \sqrt{xy}\right) \quad \text{for } x,\ y > 0.$$

(a) Compute the Jacobian matrix of ϕ. **(2 p)**

(b) Use a linear approximation in order to approximate the value of $\phi(1.01,\ 0.99)$. **(2 p)**

Solution.

(a) The Jacobian matrix is given by

$$J_\phi(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2x} & -\frac{1}{2y} \\ \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \end{pmatrix}.$$

(b) We have $\phi(1,1) = (0,1)$ and the Jacobian matrix of ϕ at the point $(1,1)$ is given by:

$$J_\phi(1,1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Thus, the linear approximation at the point $(1,1)$ takes the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}.$$

Applying this linear approximation with $x = 1{,}01$ and $y = 0{,}99$, we obtain

$$\phi(1.01,\ 0.99) \approx (0.01,\ 1.00).$$

Answer.

(a) See the solution.

(b) $\phi(1.01,\ 0.99) \approx (0.01,\ 1.00)$

6. It has been observed that the fish in a certain fishing area is always swimming according to a certain flow that can be described by the vector field

$$\mathbf{F}(x, y, z) = (1, -2xy, 2xz) \quad \text{[kilogramme fish per minute and unit area]},$$

where x and y are the geographic coordinates and $z \leq 0$ the depth.

(a) Give the equations that determine the flow lines of the vector field. **(1 p)**

(b) A plane quadratic net can be either fixed between the points

$$A: \quad (0,0,0),\ (0,1,0),\ (0,1,-1),\ (0,0,-1),$$

or

$$B: \quad (0,1,0),\ (1,1,0),\ (1,1,-1),\ (0,1,-1).$$

Which of the alternatives must be chosen in order to maximise the catch, and how much fish will have been caught with that particular net if it has been left out for the duration of 5 hours? **(3 p)**

Solution.

(a) The flow lines are determined by the equations

$$dx = -\frac{dy}{2xy} = \frac{dz}{2xz}.$$

(b) We assume that a fish gets caught in either of the fishing nets independent of the direction with which it enters the net. This means that, in order to compute the total amount of fish that gets caught in a net per unit of time, it is *not* the (net) flow of $\mathbf{F}$ through the net that needs to be computed, i.e. not $\mathbf{F} \cdot \hat{\mathbf{N}}$, but rather the integral of $\left|\mathbf{F} \cdot \hat{\mathbf{N}}\right|$ over the net. Here, $\hat{\mathbf{N}}$ denotes, as usual, a unit normal field to the surface, i.e. the net. (It will become clear in the course of this solution that $\mathbf{F} \cdot \hat{\mathbf{N}}$ does not change sign on either of the nets. This means that we obtain the correct answer even if we ignore the absolute value signs.)

The net A is the subset of the plane $x = 0$ on which $0 \leq y \leq 1$ and $-1 \leq z \leq 0$. Thus, the constant vector field $\hat{N}_A = (1, 0, 0)$ forms a unit normal field to A. Let Ψ_A be the amount of fish that gets caught in net A per minute. Then we have

$$\Psi_A = \iint_A \left|\mathbf{F} \cdot \hat{\mathbf{N}}\right| dS = \iint_A |(1, -2xy, 2xz) \cdot (1,0,0)|\, dS = \int_0^1 \int_{-1}^0 1\, dz dy = 1$$

The net B is the subset of the plane $y = 1$ on which $0 \leq x \leq 1$ and $-1 \leq z \leq 0$. Thus, the constant vector field $\hat{N}_B = (0, 1, 0)$ is a unit normal field for B. Let Ψ_B be the amount of fish that gets caught in the net B per minute. Then we have

$$\begin{aligned}\Psi_B &= \iint_B \left|\mathbf{F} \cdot \hat{\mathbf{N}}\right| dS = \iint_A |(1, -2xy, 2xz) \cdot (0,1,0)|\, dS = \int_0^1 \int_{-1}^0 2xy|_{y=1}\, dz dx \\ &= \int_0^1 \int_{-1}^0 2x\, dz\, dx = \int_0^1 2x\, dx = x^2|_0^1 = 1.\end{aligned}$$

Thus, both alternatives are equally good, both lead to a catch of 1kg fish per minute, or, in other words, 300kg fish per five hours.

Answer. (a) $dx = -\frac{dy}{2xy} = \frac{dz}{2xz}$ (b) Both alternatives lead to an equally large catch of 300kg fish within 5 hours.

DEL C

7. Let f be the function defined by

$$f(x, y, z) = x + y + z$$

on the domain

$$\mathcal{D}(f) = \left\{(x, y, z) \in \mathbb{R}^3 : xy = 1 \text{ och } x^2 + y^2 + z^2 = 4\right\}.$$

Determine the largest and the smallest value that f assumes on the domain $\mathcal{D}(f)$.

(4 p)

Solution. The domain $\mathcal{D}(f)$ consists of closed curves lying on the sphere of radius 2 about the origin. This is a closed and bounded set. Since the function f is continuous, we therefore know that it attains both its minimum and maximum. We will apply Lagrange's method in order to find these.

The domain of the function $f(x, y, z) = x + y + z$ can be described by the following two conditions (i.e. constraints):

$$g(x, y, z) = xy = 1$$

and

$$h(x, y, z) = x^2 + y^2 + z^2 = 4.$$

Minima and maxima of f occur only at such points at which $\operatorname{grad} f$ is a linear combination of $\operatorname{grad} g$ and $\operatorname{grad} h$, that is at points where

$$\operatorname{grad} f = \lambda \operatorname{grad} g + \mu \operatorname{grad} h$$

for some choice of real numbers λ and μ. This condition yields

$$(1, 1, 1) = \lambda(y, x, 0) + \mu(2x, 2y, 2z),$$

i.e.

$$\begin{aligned} 1 &= \lambda y + 2\mu x \\ 1 &= \lambda x + 2\mu y \\ 1 &= 2\mu z, \end{aligned}$$

and these last three equations need to be satisfied in addition to the constraints

$$xy = 1 \qquad \text{and} \qquad x^2 + y^2 + z^2 = 4.$$

Subtracting the second equation from the first yields

$$0 = \lambda y + 2\mu x - \lambda x - 2\mu y = 2\mu(x - y) - \lambda(x - y) = (2\mu - \lambda)(x - y),$$

so that either $2\mu = \lambda$ or $x = y$ holds true. If $2\mu = \lambda$, none of these numbers can be zero. In this case, we deduce from the first and the third of the above equations that

$$\lambda(x + y) = \lambda y + 2\mu x = 1 = 2\mu z = \lambda z,$$

and, hence,

$$z = x + y.$$

If we add together the constraint $h = 4$ and twice the constraint $g = 1$, we obtain

$$x^2 + 2xy + y^2 + z^2 = 4 + 2,$$

which, upon inserting the previous equation, becomes

$$(x + y)^2 + (x + y)^2 = 6,$$

and, hence,

$$x + y = \pm\sqrt{3}.$$

Combining this with $xy = 1$ yields

$$0 = x^2 \pm \sqrt{3}x + 1 = \left(x \pm \frac{\sqrt{3}}{2}\right)^2 - \frac{3}{4} + 1 = \left(x \pm \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4},$$

which is a contradiction. Thus, there are no solutions if $2\mu = \lambda$. If, on the hand, $x = y$, then it follows from $xy = 1$ that

$$x = y = \pm 1.$$

Inserting this into $x^2 + y^2 + z^2 = 4$, we obtain $2 + z^2 = 4$, i.e.

$$z = \pm\sqrt{2}.$$

We have found four points which could be minima or maxima. The values that f takes at these points are:

$$\begin{aligned} f(1, 1, \sqrt{2}) &= 2 + \sqrt{2} \\ f(1, 1, -\sqrt{2}) &= 2 - \sqrt{2} \\ f(-1, -1, \sqrt{2}) &= -2 + \sqrt{2} \\ f(-1, -1, -\sqrt{2}) &= -2 - \sqrt{2}, \end{aligned}$$

amongst which the largest and smallest values are:

$$f(1, 1, \sqrt{2}) = 2 + \sqrt{2} \qquad \text{and} \qquad f(-1, -1, -\sqrt{2}) = -2 - \sqrt{2}.$$

Answer. The largest value is $2 + \sqrt{2}$ and the smallest value is $-2 - \sqrt{2}$.

8. Green's theorem is an important tool for computing line integrals.
   (a) State Green's theorem. Do not forget to include all conditions. **(1 p)**
   (b) Prove Green's theorem in the special case where the curve is given as the positively oriented boundary of the unit square, i.e. of the domain defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1$. **(3 p)**

Solution. A.

(a) Let E be a regular, closed domain in the xy-plane whose boundary γ consists of one or more piecewise smooth, simple, closed curves that are positively oriented with respect to E. Let $\mathbf{F} = (P(x,y), Q(x,y))$ be a smooth vector field on E. Then,

$$\int_\gamma (P\,dx + Q\,dy) = \iint_E \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dxdy.$$

(b) Call the unit square E and its positively oriented boundary γ. Then,

$$\begin{aligned}\iint_E -\frac{\partial P}{\partial y}\,dxdy &= \int_0^1 \left(\int_0^1 -\frac{\partial P}{\partial y}\,dy\right)\,dx \\ &= \int_0^1 \left[-P(x,y)\right]_{y=0}^{y=1}\,dx \\ &= \int_0^1 -P(x,1)\,dx - \int_0^1 -P(x,0)\,dx \\ &= \int_1^0 P(x,1)\,dx + \int_0^1 P(x,0)\,dx \\ &= \int_\gamma P\,dx,\end{aligned}$$

since γ is positively oriented and since the contribution of the vertical parts is zero. A similar argument shows that

$$\iint_E \frac{\partial Q}{\partial x}\,dxdy = \int_\gamma Q\,dy.$$

Combining these two parts completes the proof of the special case.

Answer. See the solution.

9. If two points (x_1, y_1) and (x_2, y_2) are chosen uniformly at random within the unit circle we can compute the average area of the triangle given by the two points together with the origin as
$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1\int_0^1 \frac{r_1 r_2}{2}\left|\sin(\theta_1-\theta_2)\right| \, r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2.$$
Compute this average area. **(4 p)**

Solution.

$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1\int_0^1 \frac{r_1 r_2}{2}\left|\sin(\theta_1-\theta_2)\right| \, r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2$$
$$= \frac{1}{2\pi^2}\left(\int_0^{2\pi}\int_0^{2\pi}\left|\sin(\theta_1-\theta_2)\right| d\theta_1 d\theta_2\right)\left(\int_0^1 r_1^2\,dr_1\right)\left(\int_0^1 r_2^2\,dr_2\right)$$

In order to take care of the absolute sign, we can make a change of variables so that $s = \theta_1 - \theta_2$ and $t = \theta_2$. We get the Jocobian to be 1 and hence $dsdt = d\theta_1 d\theta_2$.

$$\int_0^{2\pi}\int_0^{2\pi}\left|\sin(\theta_1-\theta_2)\right| \, d\theta_1 d\theta_2 = \int_0^{2\pi}\int_{-t}^{2\pi-t}\left|\sin(s)\right| \, dsdt = [\sin(s) \text{ har period } 2\pi]$$
$$= \int_0^{2\pi}\int_0^{2\pi}\left|\sin(s)\right| \, dsdt = 2\pi\int_0^{\pi}\sin(s)\,ds + 2\pi\int_\pi^{2\pi} -\sin(s)\,ds$$
$$= 2\pi\left[-\cos(s)\right]_0^\pi + 2\pi\left[\cos(s)\right]_\pi^{2\pi}$$
$$= 2\pi(-(-1)-(-1)+1-(-1)) = 8\pi.$$

The last two factors are equal and are given by
$$\int_0^1 r^2\,dr = \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3}.$$

In conclusion we get the average area as
$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1\int_0^1 \frac{r_1 r_2}{2}\left|\sin(\theta_1-\theta_2)\right| \, r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2 = \frac{1}{2\pi^2}\cdot 8\pi\cdot\left(\frac{1}{3}\right)^2 = \frac{4}{9\pi}.$$

Answer. The average area is $4/(9\pi)$ area unis.