Assesed homework 2nd set.

The second set of homework is **due on the 9th December (preliminary date).** The date can change depending on how fast we progress.

Assignemnt 1: Let $g \in L^1(\mathbb{R})$ and $f(x)$ be a measurable function such that

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq g(x) \tag{1}$$

for all $|h| < 1$ (and $h \neq 0$).

1. Show that f is continuous on $\mathbb{R}$.

2. Show that f is bounded on $\mathbb{R}$ and give an upper bound on $f(x)$ under the assumption that $f(0) = 0$.

3. **(Voluntary assignment)** Is the same true if we, instead of (1), assume that

$$\int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx \leq C,$$

for some C independent of h, $|h| < 1$ and $h \neq 0$?

Assignemnt 2: Let

$$J(f) = \int_0^1 x^3 |f'(x)|^2 dx$$

be defined on

$$K = \{g \in W^{1,2}(0,1);\ g(0) = 0 \text{ and } g(1) = 1\}.$$

Show that we may find a sequence $f_j \in K$ such that

1. $$\lim_{j \to \infty} J(f_j) \leq \inf_{g \in K} J(g),$$

2. $f_j \to f_0$ strongly in $L^2(0,1)$ and $f_0 \in W^{1,2}(0,1)$.

3. $f_0 \notin K$.

[Hint] *Can you arrange it so that $f_0(0) \neq 0$?*

Assignemnt 3: Imagine that a gifted student asks you the following:

"I have heard that there exists a minimizer of

$$\int_{\mathcal{D}} |\nabla u(\mathbf{x})|^{3/2} + |u(\mathbf{x})|^q d\mathbf{x}$$

where $\mathcal{D}$ is a domain in $\mathbb{R}^2$ and $q > 0$ and $u(x) = f(x)$ on $\partial\mathcal{D}$. It was some such result, I can't really remember all the assumptions or exactly what q was. I certainly don't know how to prove it. Could you help me out and explain to me how such result is proved? I think that I remember that there was some sort of condition on q besides $q > 0$. Do you think that you could give a least upper bound on all $q > 0$ such that a minimizer exists?"

Since you are such a kind hearted human being you have decided to write a few (maybe ≈ 3) pages of informal explanation of how such a result is proved. Your explanations should include:

1. The general strategy of how such a problem is approached with the most important theory from the course mentioned.

2. You are not required to prove anything. But explain why the theory looks the way it does.
3. Try to give the idea of some proofs or at least examples of why the theorems are needed.
4. You are not required to prove that your least upper bound of q is a least upper bound. But indicate what theorem is needed and say something about why there is a restriction on q, maybe by giving an illustrative example.

Naturally, this assignment will be read by your dear lecturer (and assessed!). Therefore, you need to show that you have read the course material and thought it through carefully.