PROBLEM SET B

1. Consider the Poisson equation:

$$\begin{aligned} -\Delta u(x) &= 1 \quad x \in \Omega \\ u(x) &= 0 \quad x \in \Gamma \end{aligned}$$

   a) Derive a variational formulation for the Poisson equation: Find $u \in V$ such that

   $$a(u, v) = L(v), \qquad \text{for all } v \in V$$

   Define the Hilbert space $V = H_0^1(\Omega)$, and its associated norm $\| \cdot \|_V$, and define the bilinear form $a : V \times V \to \mathbb{R}$ and the linear form $L : V \to \mathbb{R}$.

   a) Prove that there are constants $\kappa_1 > 0$, κ_2, κ_3, such that for all $v, w \in V$:

   $$a(v, v) \geq \kappa_1 \|v\|_V^2, \quad |a(v, w)| \leq \kappa_2 \|v\|_V \|w\|_V, \quad |L(v)| \leq \kappa_3 \|v\|_V$$

   and that there exists a unique solution u to the variational problem.

   c) Formulate an abstract Galerkin method with solution U, in terms of the bilinear and linear forms, using a finite dimensional subspace $V_h \subset V$.

   d) Prove that

   $$\|u - U\|_V \leq \frac{\kappa_2}{\kappa_1} \|u - v\|_V, \qquad \text{for all } v \in V_h$$

2. For $a(x) > 0$ and $c(x) \geq 0$, consider the problem:

$$-(a(x)u'(x))' + c(x)u(x) = f(x), \quad x \in (0, 1), \quad u(0) = u(1) = 0$$

   a) Formulate the cG(1) method for the problem (FEM with a continuous piecewise linear approximation on a subdivision $\mathcal{T}_h$ of (0,1)).

   b) Prove the a posteriori error estimate:

   $$\int_0^1 (u - U)\xi \; dx \leq C_i \|h^2 R(U)\|_{L_2(0,1)} \|\varphi''\|_{L_2(0,1)}$$

   where U is the cG(1) solution, and φ is the solution to the dual problem:

   $$-(a(x)\varphi'(x))' + c(x)\varphi(x) = \xi(x), \quad x \in (0, 1), \quad \varphi(0) = \varphi(1) = 0$$

3. Consider the heat equation:

$$\begin{aligned} \dot{u}(x, t) - \Delta u(x, t) &= f(x, t), & (x, t) &\in \Omega \times (0, T] \\ u(x, t) &= 0, & (x, t) &\in \Gamma \times (0, T] \\ u(x, 0) &= u^0(x), & x &\in \Omega \end{aligned}$$

   Prove that:

   $$\frac{d}{dt} \|u\|^2 + \|\nabla u\|^2 \leq C \|f\|^2$$

   where $C > 0$ is the constant in the Poincaré-Friedrich inequality.