F2

Last lecture we constructed

$\mathbb{R}$ = set of all cuts $A|B$.

Today we will show that $\mathbb{R}$ works well with limits and discuss different infinity's.

Remember that

$\lim_{n\to\infty} a_n = a$ if $\forall \varepsilon > 0 \ \exists N_\varepsilon > 0$ s.t.

$$n > N_\varepsilon \Rightarrow |a - a_n| < \varepsilon .$$

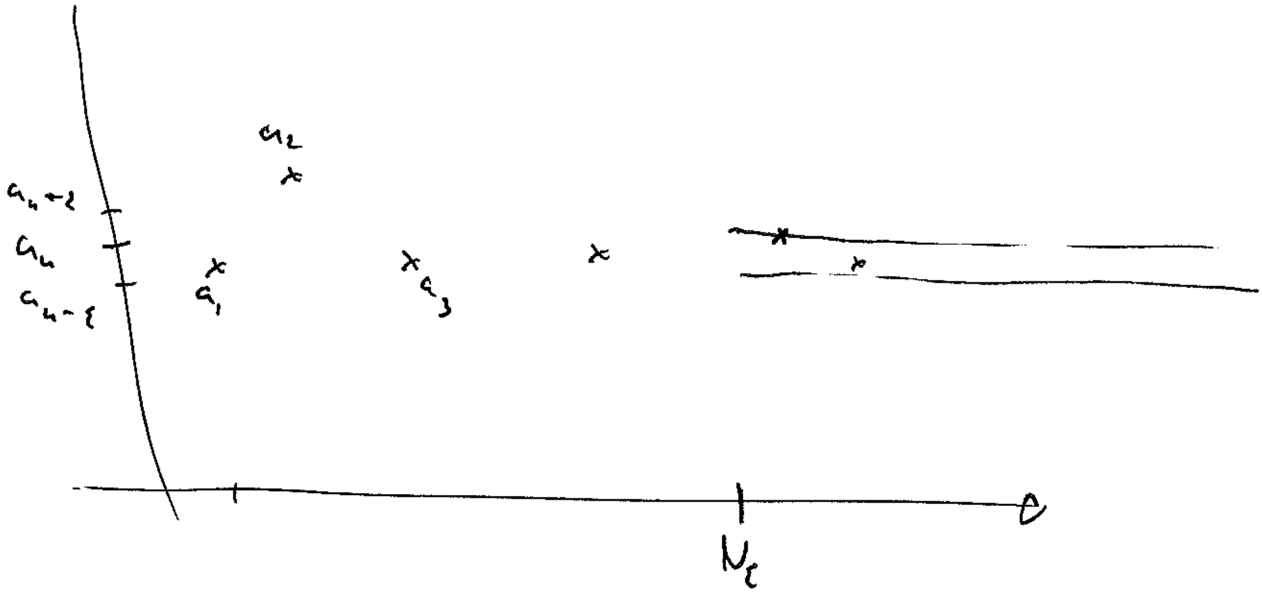
The main problem with this definition is that we have to know what a is in order to use it.

Def We say that a_n is a Cauchy-sequence if for every $\varepsilon > 0$ $\exists N_\varepsilon$ s.t.

$$n, m > N_\varepsilon \Rightarrow |a_n - a_m| < \varepsilon$$

Thm: $\mathbb{R}$ is complete w.r.t. Cauchy sequences. That is if a_n is Cauchy then $\exists a \in \mathbb{R}$ s.t. $a_n \to a$.

Proof: We need to identify the limit, in order to do that we need to use the l.u.b.-property.



Denote $X = \{x;\ x > a_n \text{ for infinitely many } n\}$

and $a = \text{l.u.b.}(X)$.

To see that a exsists we need to show that

$X \neq \emptyset$ and X is bounded from above.

But $\exists N_1$ s.t.

$$n > N_1 \Rightarrow \quad |a_{N_1} - a_n| < 1 \quad \Rightarrow \underbrace{a_{N_1}-1 < \overbrace{a_n < a_{N_1}+1}^{(*)}}_{(**)}$$

From $(*)$ we see that $x \leq a_{N_1}+1$ for all $x \in X$ (since only N_1 numbers are smaller than $a_{N_1}+1$)

From $(**)$ we see that $a_{N_1}-1 \in X$

since all $a_n > a_{N_1}-1$ when $n > N_1$.

We need to show that $a_n \to a$.

To do this we notice that $a - \varepsilon/2$ is not an upper bound of X (for $\varepsilon > 0$).

Therefore there exists ∞ many $a_n > a - \varepsilon/2$.
and only finitely many $a_n < a + \varepsilon/2$
pick one such n, say $n_{\varepsilon/2}$, s.t. $n_{\varepsilon/2} > N_{\varepsilon/2}$.

Then for any

$n > n_{\varepsilon/2}$ $$|a - a_n| = |(a - a_{\varepsilon/2}) + (a_{\varepsilon/2} - a_n)| \leq$$

$$\leq \underbrace{|a - a_{\varepsilon/2}|}_{< \varepsilon/2} + \underbrace{|a_{\varepsilon/2} - a_n|}_{< \varepsilon/2} < \varepsilon.$$

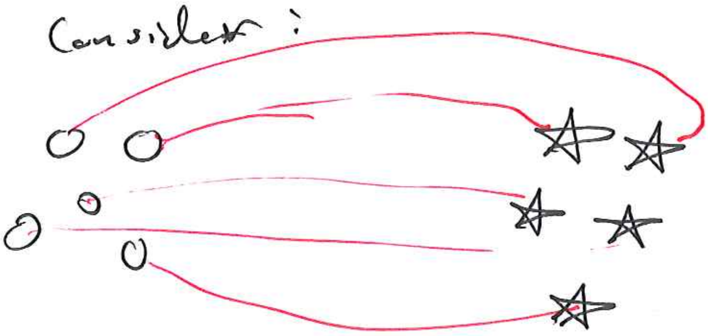
since $n > n_{\varepsilon/2} > N_{\varepsilon/2}$

Cardinality.

~~The~~ Two things make analysis analysis 1) limits
2) ∞

In order to understand infinity we need to be very careful about counting.

Consider:



The circles and stars are equally many since we can pair them up (say draw a line from one to the other). This makes sense if we have a few objects. But if we have many (infinitely many) then this method becomes impractical. But we can do it in principle if we interpret "the line" as a function.

<u>Def:</u> Låt $f: A \mapsto B$ vara en funktion från A till B.
Då säger vi att f är

injektiv om $f(x) = f(y) \Rightarrow x = y$

surjektiv om för alla $y \in B$ det existerar ett $x \in A$ så att $f(x) = y$

bijektiv om f är injektiv och surjektiv.

<u>Def:</u> Vi säger att ~~A~~ två mängder A och B har samma kardinalitet om det finns en funktion $f: A \mapsto B$ som är bijektiv. Då $A \sim B$

Remarks: a) If $A \sim B$ then $B \sim A$ since
$f^{-1}: B \longmapsto A$ is a bijection

b) $A \sim A$ (identity mapping)

c) $A \sim B$ and $B \sim C \Rightarrow A \sim C$
(since $f: A \longmapsto B$ and $g: B \longmapsto C$ gives rise to $g \circ f: A \longmapsto C$ which is a bijection).

Therefore the relation $\sim$ divides "all sets" into equivalence classes.

Def * A is finite if $A = \emptyset$ or
$A \sim \{1, 2, \dots, n\}$

* A is infinite if A isn't finite
* A is denumerable if $A \sim \mathbb{N}$
* A is countable if finite or denumerable
* uncountable if not countable.

Examples: 1) The even natural numbers ~~are~~ have the same cardinality as $\mathbb{N}$.

~~$\{0, 2, 4, 6$~~

Let $f(x) = \frac{x}{2}$. ~~then~~

2) The set $\{2, 3, 4, 5 \dots\} \sim \mathbb{N}$
Let $f(x) = x - 2$.

The next question is if there are any uncountable sets.

<u>Thm</u>: $\mathbb{R}$ is uncountable.

<u>Proof</u>: We will argue by contradiction and assume that $\mathbb{R}$ is countable. That is that ~~R~~ there exists an $f(n): \mathbb{N} \longmapsto \mathbb{R}$ that is bijective.

n	$f(n)$
0	3.4 7 6 2 9 5 3 8 7
1	0.6 3 1 1 2 3 9 9 5
2	0.2 9 9 9 0 1 4 5 2
3	2.6 3 5 4 . . .
4	−1.2 3 4 5 1

etc.

In order to show that f isn't bijective it is enough to find one real number x s.t. $f(n) \neq x$ for all $n \in \mathbb{N}$ (that is f not surjective.

Consider the diagonal decimals
4 3 9 4 1 —

and construct a new real number by changing all decimals to 1 except 1 which we change to 2. So

$\cancel{0}.\ 4\ 3\ 9\ 4\ 1$

$x = 0.\ 1\ 1\ 1\ 1\ 2 \dots$

Then x is not equal to any number on the list. Thus $f(n) \neq x$ for all n. $\blacksquare$

For any set A let us denote by $P(A)$ the set of all subsets of A.

<u>Example</u> If $A = \{0, 2, 5\}$ then

$$P(A) = \left\{ \phi, \{0\}, \{2\}, \{5\}, \{2,5\}, \{0,5\}, \{0,2\}, \{0,2,5\} \right\}$$

$$P(\mathbb{R}) = \{ \text{the set of subsets of } \mathbb{R} \}.$$

<u>Thm</u>: ~~The se~~ For any set A, $P(A)$ has larger cardinality than A.

<u>Proof</u>: Assume that there is a bijection $f: A \mapsto P(A)$ and construct the set

$$C = \left\{ x \in A;\ x \notin f(x) \right\}$$

Then there is no $x \in A$ s.t. $f(x) = C$

If ~~there exists~~ such a set exsists then

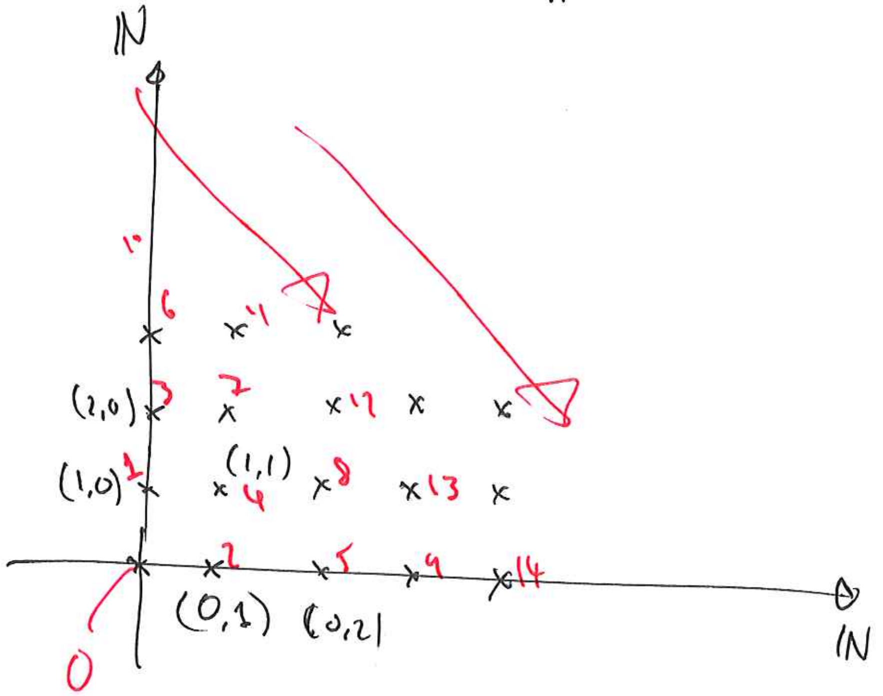
$\cancel{\to}$ if $x \in f(x) = C$ then $x \notin f(x)$ cont. and reverse...

Corollary: There is no set with the largest cardinality.

Let us deduce some more Theorems about cardinalities.

Thm: $\mathbb{N} \times \mathbb{N} = \{(a,b) ;\ a,b \in \mathbb{N}\}$ is countable.

Proof: We can draw $\mathbb{N} \times \mathbb{N}$



$\square$

Corollary: $\mathbb{N}^k$ is countable for any $k \in \mathbb{N}$.

Corollary: $\mathbb{Q}$ is countable

Corollary: If A_k is a countable set for $k = 0, 1, 2, \ldots$ then $\bigcup A_k$ is countable

Proof: If A_k is countable then $A_k = \{a_{k0}, a_{k1}, \ldots\}$

Thus $\bigcup A_k = \{a_{kl} ;\ k, l \in \mathbb{N}\} \sim \mathbb{N}^2$.

Since the cardinality of $P(A)$ is larger than the cardinality of A there is no largest cardinality. Is there a smallest one?

<u>Thm</u>: If A is infinite then A contains a denumerable subset

<u>Proof</u>: Since A is infinite $\exists x_0 \in A$.

Also $A \setminus \{x_0\}$ cannot be empty so there exist $x_1 \in A \setminus \{x_0\}$

$\vdots$

Induction $A \setminus \{x_0, x_1, \ldots, x_k\}$ not empty

so $\exists x_{k+1} \in A$ $\vdots$

$\blacksquare$

Remark: This theorem is based on the axiom of Choice.

Let us prove one last, and important Theorem.

Thm: Every open interval, (a,b), contains a real and a rational number.

Proof: Let us assume that $a>0$, and that $b-a=\varepsilon>0$. Then we can choose $j\in\mathbb{N}$ such that $\frac{1}{j}<\varepsilon$ (or $\frac{\sqrt{2}}{j}<\varepsilon$) and consider the

sequence $a_k=\frac{k}{j}$. Since $a_k\to\infty$ there exist a K s.t. $a_K>a$ and also $a_0<a$. Therefor there exist a largest k, say k_0, s.t. $a_{k_0}\le a$ and $a < a_{k_0+1}=a_{k_0}+\frac{1}{j}<a_{k_0}+\varepsilon<b$. Thus $a_{k_0+1}=\frac{k_0+1}{j}\in\mathbb{Q}$ is contained in (a,b).

$\blacksquare$

Corollary: $\mathbb{Q}$ is dense in $\mathbb{R}$. That is for each $r\in\mathbb{R}$ there exsists a sequence $q_k\in\mathbb{Q}$ s.t. $q_k\to r$.

Proof: choose $q_k\in\left(r-\frac{1}{k},r+\frac{1}{k}\right)$ and $q_k\in\mathbb{Q}$.

$\blacksquare$

The corollary is very important because $\mathbb{Q}$ is countable, which is a nice small infinity, and $\mathbb{R}$ is uncountable which is an infinity that is much more difficult to handle. In some situations we are able to prove something for $\mathbb{Q}$ that we can't prove for $\mathbb{R}$. But since $\mathbb{Q}$ is dense in $\mathbb{R}$ that also says something about $\mathbb{R}$.