Lecture 1.

Pugh states that it is reasonable to "accept" facts about $\mathbb{N}$, $\mathbb{Z}$ & $\mathbb{Q}$. But I want to say something about $\mathbb{N}$, $\mathbb{Z}$ & $\mathbb{Q}$ before we start to discuss $\mathbb{R}$.

$\mathbb{N}$ the natural numbers: $0, 1, 2, \ldots$

What are the natural numbers? To answer this we would have to dig deep into the foundations of mathematics. Therefore we will not answer that question. But I feel that, at this level, I at least owe you to give an idea of how $\mathbb{N}$ is constructed.

At the most basic level $\mathbb{N}$ is about counting, going through the numbers in sequence. Let us therefore start with 0 and then consider the successor of 0, $S(0)$, the successor of $S(0)$ will be $S(S(0))$ e.t.c.

We assume that

A1) The number 0 exsists (is a natural number)

A2) If n is a natural number then $S(n)$ is one. [axiom of infinity]

A3) There is no natural number n s.t. $S(n) = 0$

A4) If $S(n) = S(m)$ then $n = m$

A5) If a property holds for 0 and if, for any natural number n, it holds for n then it holds for $S(n)$ then it holds for all natural numbers.

We denote $0 = 0$, $1 = S(0)$, $2 = SS(0)$, $3 = SSS(0)$, etc.

Some remarks.

1) By being very explicit with what we assume we can be very sure that no unwanted assumptions sneak into mathematics. This is the core of the mathematical method

2) We may interpret $S(0)$ as 1 $S(S(0))$ = as 11 etc. $S(1111) = \underbrace{1111\ 1}_{\text{one more line}}$. Clearly A1, A2, A3, A4 holds. A5 is just the principle of induction.

A3 states in essence that there are infinitely many natural numbers

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The natural numbers are not interesting without us being able to do something with them. Let us define $+$, $\times$ and exponention s.t.

A1	$x + 0 = x$	A2	$x + Sy = S(x+y)$
M1	$x \cdot 0 = 0$	M2	$x \cdot Sy = x \cdot y + x$
E1	$x^0 = 1 = S(0)$	E2	$x^{Sy} = x^y \cdot x$

These are all we need to define $+$, $\times$ and exp.

Say $x = SS(0)$ and $y = SSS(0)$ then

$$x + y = SS(0) + S(SS(0)) = \{A2\} = S(SS(0) + S(S(0))) = \{A2\} =$$
$$= SS(SS(0) + S(0)) = \{A2\} = SSS(SS(0) + 0) = \{A1\} =$$
$$= SSS(SS(0)) = SSSSS(0).$$

With these definitions we can prove normal rules of calculation

Thm: i) $x+y = y+x$

ii) $x(y+z) = xy + xz$

etc.

proof: [of i)]

Step 1: $0+x = x$

Let $S = \{\text{all } x \text{ such that } 0+x = x\}$

then $0 \in S$ since $0+0=0$ by A1

if $x \in S$ then $Sx \in S$ since

$0+Sx = \{A2\} = S(0+x) = \{x \in S\} = S(x)$.

Step 2 $x+Sy = Sx+y$

We prove this by induction on y

Basis: $x+S(0) = \{A2\} = S(x+0) = \{A1\} = S(x)$

Induction: Assume that $x+Sy = Sx+y$ then

$x+SSy = \{A2\} = S(x+Sy) = S(Sx+y) = \{A2\} = Sx+Sy$

Step 3 $x+y = y+x$

Again by induction on y

Basis ($y=0$ case) $x+0 = \{A1\} = x = \{\text{step } 1\} = 0+x$

Induction: Assume $x+y = y+x$ then

$x+Sy = \{A2\} = S(x+y) = S(y+x) = \{A2\} = y+Sx = \{\text{Step } 2\} = Sy+x$.

In this way we can construct, and prove, everything for $\mathbb{N}$ that we are familiar with.

Say $x > y$ if there exist $z \in \mathbb{N}$ s.t $z \neq 0$ and $x = y + z$.
Then we write $z = x - y$.

The integers $\mathbb{Z}$

To construct the integers $\mathbb{Z}$ we consider pairs (n, m), $n, m \in \mathbb{N}$ and $n = 0$ and/or $m = 0$

positive negative

we may define

$$(x,y) + (n,m) = \begin{cases} (x+n, 0) & \text{if } y = m = 0 \\ (0, y+m) & \text{if } x = n = 0 \\ (x-m, 0) & \text{if } y = n = 0 \quad \text{if } x > m \\ (0, m-x) & \text{— '' —} \quad m > x \\ \text{etc.} & \end{cases}$$

of course we write $(0,m) = -m$ and $(n,0) = n$.

There are many things to prove but with ~~those~~ this definition we may construct $\mathbb{Z}$.

The rational numbers $\mathbb{Q}$

We define $\mathbb{Q}$ = all pairs (n, m) $\quad n \in \mathbb{Z}$, $m \in \mathbb{N}$, $m \neq 0$

Then we can define

$(x,y) + (n,m) = (xm + yn, ym)$ etc.

$(n,m) > (x,y)$ if $ny > xm$ etc.

The construction of $\mathbb{R}$. (This Course)

What is the difference between $\mathbb{R}$ and $\mathbb{Q}$?

We need one more axiom

Completeness Axiom. (Least upper bound property)
~~Let~~ If S is a nonempty subset of $\mathbb{R}$ s.t.

a) S is bounded from above:
$\exists M \in \mathbb{R}$ s.t. $s \leq M$ for all $s \in S$
M is an upper bound.

b) $S \neq \emptyset$

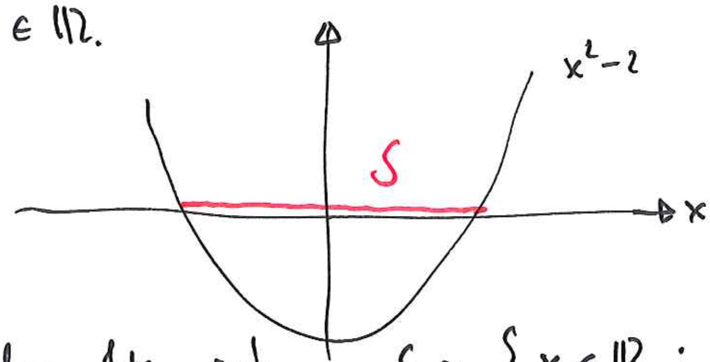
Then S has a least upper bound $m = \text{lub}(S)$

i) m is an upper bound: $s \leq m$ for all $s \in S$

ii) if M is an upper bound then $m \leq M$

Why is this important.

Example: (roots of equations) Does $x^2 - 2 = 0$ have a solution? Clearly $\pm\sqrt{2}$ solves the equation but $\pm\sqrt{2} \notin \mathbb{Q}$ so we need to show that $\pm\sqrt{2} \in \mathbb{R}$.



consider the set $S = \{x \in \mathbb{R};\ x^2 < 2\}$

Then the least upper bound of S exists in $\mathbb{R}$. $\sqrt{2} = \text{l.u.b}(S) \in \mathbb{R}$.

Example: The number e.

If we define $e = \sum_{k=1}^{\infty} \frac{1}{k!}$, ~~then~~ how do we know that e exsists?

Clearly the numbers $e_m = \sum_{k=1}^{m} \frac{1}{k!}$ exsists in $\mathbb{Q}$. Moreover for each $m > 2$

$$e_m = \sum_{k=0}^{m} \frac{1}{k!} < 1+1+\sum_{k=2}^{m} \frac{1}{k\cdot(k-1)} = 1+1+\sum_{k=2}^{m}\left(\frac{1}{k-1}-\frac{1}{k}\right) =$$

$$= 1+1+\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\cancel{\dots}+\left(\cancel{\frac{1}{m-2}}-\cancel{\frac{1}{m-1}}\right)+\left(\cancel{\frac{1}{m-1}}-\frac{1}{m}\right)$$

$$= 3-\frac{1}{m} < 3$$ so the set

$S = \{e_1, e_2, e_3, e_4, \dots\}$ is nonempty and bounded from above, therefore there exists a $l.u.b(S) = u$

since $l.u.b(S)$ is the least upper bound $u-\varepsilon$ is not an upper bound: $\exists\, e_N$ s.t.

$u-\varepsilon < e_N \leq u$, but e_m is increasing

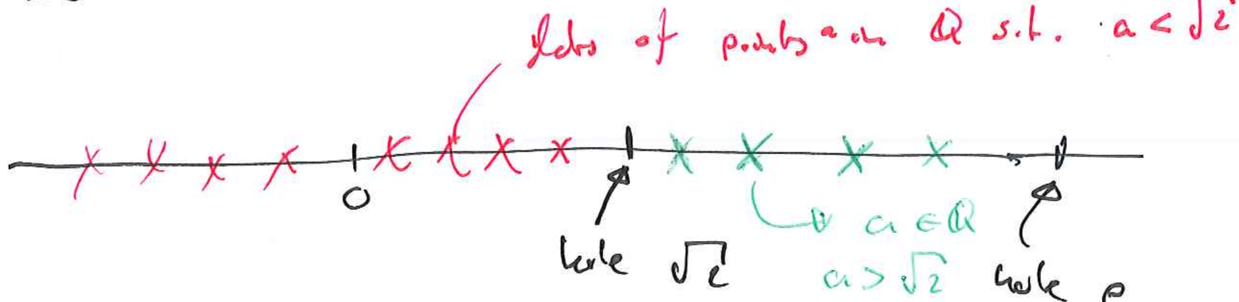
so for $m > N$ $u-\varepsilon < e_N < e_m \leq u$

$\Rightarrow$ if $m > N$ then $|e_m - u| < \varepsilon$.

So $e_m \to u$ and therefore $\sum_{k=0}^{\infty} \frac{1}{k!}$ exsists.

How do we construct something that satisfies the l.u.b-property?

We have the rational line $\mathbb{Q}$



and we want to characterize and fill in the holes. The idea is that even if there isn't a rational number where we have a hole still the hole is characterized by the rational points to the left and right of the hole.

So the hole at $\sqrt{2}$ should be characterized by all $a \in \mathbb{Q}$ s.t. $a < 0$ or $a^2 < \sqrt{2}$

Definition: We say that $A, B \subset \mathbb{Q}$ is a cut if

a) $A \cup B = \mathbb{Q}$

b) if $a \in A$ & $b \in B$ then $a < b$

c) A doesn't contain a largest element.

we write $A|B$

Definition: A real number is a cut. $\mathbb{R}$ is the set of all real numbers

Thm: $\mathbb{R}$ satisfies the least upper bound property.

~~Proof:~~

In order to prove this we need to first define $<$ on $\mathbb{R}$

Def. We say that $x < y$ for $x, y \in \mathbb{R}$ if $x = A_x | B_x$ $y = A_y | B_y$ and $A_x \subset A_y$, $A_x \neq A_y$.

proof: Let S be a set of real numbers that is bdd from above: $\exists\, A|B$

If $x = A_x | B_x \in S$ then $A_x \subset A$.

Consider the real number

$$u = \underbrace{\bigcup_{x\in S} A_x}_{A_u} \Big| \underbrace{\Big(\mathbb{Q} \setminus \bigcup_{x\in S} B_x\Big)}_{B_u}$$

Then

$A_u \subset A$ since if $q \in A_u$ then $q \in A_x \subset A$ for some x. Also $A_x \subset A_u$ for all $x \in S$

so u is an upper bound for S.

To see that u is the least upper bound we notice that if v is another upper bound then $A_x \subset A_v$ for all $x \in S$ thus for any $q \in A_u$ ther is an A_x s.t. $q \in A_x$ and thus $q \in A_v$ so $q \in A_u \Rightarrow q \in A_v$ and therefor $A_u \subset A_v \Rightarrow u \le v$.

$\blacksquare$

Next we need to verify that we may add and multiply real numbers. This is very tedious but rather elementary.

Define $A_x|B_x + A_y|B_y = \{p+q;\ p\in A_x,\ q\in A_y\} \,\Big|\, \mathbb{Q}\setminus\{p+q\}$

$$(A_x|B_x)\bullet(A_y|B_y) = \begin{cases} \underbrace{\{a;\ a<0 \text{ or } a=p\cdot q \text{ for } p,q\geq 0\}}_{} & \text{if } x,y>0 \\ -(-x)\cdot(-y) \quad \text{commutative since is } \mathbb{Q} & \text{if } x,y<0 \\ -(-x)\cdot y & \text{if } x<0 \quad y>0 \\ -x\cdot(-y) & \text{if } x>0 \quad y<0 \\ 0 & \text{if } x=0 \text{ or } y=0 \end{cases}$$

etc.

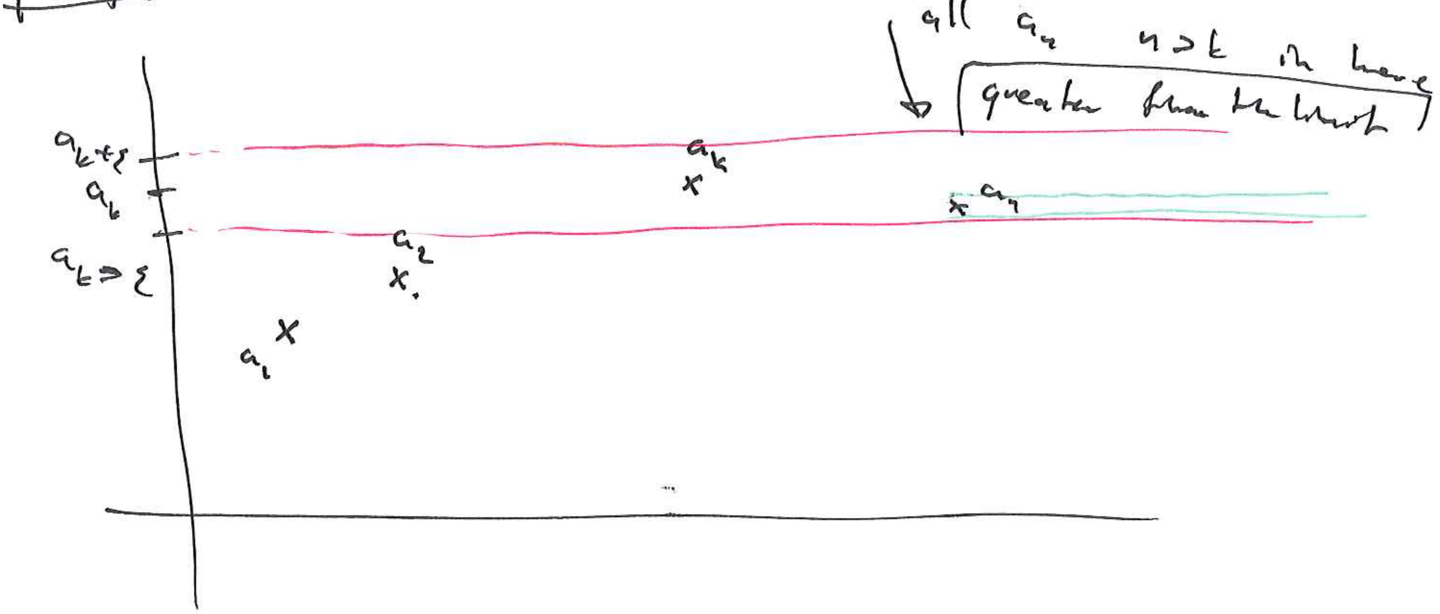
Even thoug the existence of $\sqrt{2}$ and e are important properties of $\mathbb{R}$. The most important ~~[illegible]~~ property of $\mathbb{R}$ (which implies existence of e and $\sqrt{2}$) is that ~~convergence is well~~ $\mathbb{R}$ works well with limits.

Def: We say that a sequence of numbers a_n is a Cauchy sequense if it satisfies the Cauchy condition.

$$\forall \varepsilon > \quad \exists N_\varepsilon \quad \text{s.t. if } k,n>N_\varepsilon \text{ then } |a_k-a_n|<\varepsilon$$

Theorem: $\mathbb{R}$ is complete w.r.t. Cauchy sequences in the sense that if a_n is Cauchy then $\exists a\in\mathbb{R}$ s.t. $a_n\to a$.

proof!



We need to use the l.u.b. property of $\mathbb{R}$.
To that end let $X = \{x \in \mathbb{R} \,;\, x < \text{infinitely many } a_n\}$.
We claim that X is non-empty and bounded from above.

1) X non empty $\overbrace{}^{\text{bounded}}$ since a_n is Cauchy $\exists N_1$ s.t. $|a_k - a_{N_1}| < 1$ for all $k > N_1$

~~check~~

$$\underbrace{|a_k - a_{N_1}| < 1}_{a_{N_1} - 1 < a_k < a_{N_1} + 1}$$

Thus $a_{N_1} - 1 \in X$ and $a_{N_1} + 1$ is an upper bound.

Let $a = \text{l.u.b}(X)$ then $a - \frac{\varepsilon}{2}$ is not an upper bound $\Rightarrow$ $\exists$ infinitely many a_n s.t $a_n > a - \frac{\varepsilon}{2}$ also $a + \frac{\varepsilon}{2} \notin X$ and thus there are only finitely many $a_n > a + \frac{\varepsilon}{2}$ thus for infinitely many a_n $\quad a - \frac{\varepsilon}{2} < a_n < a + \varepsilon/2 \qquad (*)$

Now pick one of the so many a_n satisfying (*) s.t. $n > N_{\varepsilon/2}$. ~~Then for all~~ call that n n_0.

Then if

$n > n_{\varepsilon/2}$ then $|a - a_n| = |a - a_{n_{\varepsilon/2}} + a_{n_{\varepsilon/2}} - a_n| \leq$

$$\leq \underbrace{|a - a_{n_{\varepsilon/2}}|}_{\substack{< \varepsilon/2 \\ \text{by } (*)}} + \underbrace{|a_{n_{\varepsilon/2}} - a_n|}_{\substack{< \varepsilon/2 \\ \text{since } n, n_{\varepsilon/2} > N_{\varepsilon/2}}} < \varepsilon .$$

Thus $a_n \to a$.

$\boxtimes$