Matrix Algebra
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Homework # 9

1. Let $A, B, C \in M_n$ and show that the equation $AXB = C$ has a unique solution $X \in M_n$ for every C if and only if both A and B are nonsingular. If either A or B is singular, show that there is a solution X if and only if $\text{rank}[B^T \otimes A] = \text{rank}\{[B^T \otimes A \quad \text{vec}(C)]\}$.

2. In, for example, the problem of estimating covariance matrices for wireless communication channels the following matrix optimization problem has appeared. Let $A \in M_{pq}$ be a given matrix. We want to find matrices $X \in M_p$ and $Y \in M_q$ whose Kronecker product approximates A as well as possible in the Frobenius norm; that is, we want to find solutions to

$$\min_{X,Y} \|A - (X \otimes Y)\|_F^2.$$

Show that this can be reformulated as a rank-one approximation problem

$$\min_{x,y} \|B - xy^*\|_F^2,$$

where x, y are vectors and B is a matrix independent of x, y. The solutions of the former problem should be given from the solutions of the latter, which can be obtained from the SVD of B.

Hints: Notice that there exists a permutation matrix P such that $\text{vec}(X \otimes Y) = P[\text{vec}(X) \otimes \text{vec}(Y)]$. Also notice that $\text{vec}(ab^T) = b \otimes a$ for vectors a, b.

3. The orthogonal projection matrix on the range space of a full rank matrix $A \in M_{m,n}$, $(m > n)$, can be written as $\Pi = AA^\dagger$. Here, $A^\dagger = (A^*A)^{-1}A^*$ is the Moore-Penrose pseudo inverse of A. Assume now that A depends on some parameter x (real scalar). Derive an expression for $d\Pi/dx$ in terms of dA/dx.

4. Let A, B, X be real matrices of appropriate dimensions. Derive

$$\frac{\partial \,\text{tr}(X^T AXB)}{\partial X}$$